Math 127B Spring 2020 Practice Midterm Answers

1. (second derivative) If \( f \) is a function defined in a neighborhood of 0 write \( E(f) = \lim_{h \to 0} \frac{f(-h) - 2f(0) + f(h)}{h^2} \) if the limit exists.

   (a) Find \( E(\frac{1}{x+2}) \).
   \[ \text{ANS: } 2 \]

   (b) Show that \( E(f + 2g) = E(f) + 2E(g) \) if \( E(f) \) and \( E(g) \) exist.
   \[ \text{ANS: If the limits exist then } E(f) + 2E(g) = \lim_{h \to 0} \frac{f(-h) - 2f(0) + f(h)}{h^2} + 2 \lim_{h \to 0} \frac{g(-h) - 2g(0) + g(h)}{h^2} = \lim_{h \to 0} \frac{f(2g)(-h) - 2(2g)(0) + (2g)(h)}{h^2} = E(2g). \]

2. For the series
   \[ f(x) = x^2 \sum_{n=0}^{\infty} 2^{-n} \cos(3^n x) \]

   (a) Determine whether \( f(x) \) is continuous in \( \mathbb{R} \).
   \[ \text{ANS: Yes.} \]
   The sum is the Weierstrass function \( W(x) \). \( W(x) \) is continuous since the summands and hence partial sums are and it is Cauchy since the partial sum from \( m \) to \( n \) is bounded by \( 2^{1-m} \). We proved theorems that Cauchy sequences converge uniformly and that uniform convergence preserves continuity. The product of the continuous function \( W(x) \) and the continuous function \( x^2 \) is continuous.

   (b) Determine whether \( f(x) \) is uniformly continuous in \( \mathbb{R} \).
   \[ \text{ANS: No.} \]
   If \( f \) is uniformly continuous then \( \forall \epsilon > 0 \exists \delta > 0 \forall |x-y| < \delta \) there is \( |f(x) - f(y)| < \epsilon \) so if \( |x-y| < n\delta \) then \( |f(x) - f(y)| < n\epsilon \). Assume this is true for contradiction. Take \( \epsilon = 1 \) and there is a \( \delta > 0 \). Take \( n > 2\pi\delta^{-1}, x = 2\pi(n+1) \) and \( y = 2\pi n \) so \( |x-y| = 2\pi < n\delta \). Compute \( f(2\pi n) = (2\pi n)^2 \cdot 2 \). Thus \( 8\pi^2(2\pi n + 1) = |f(x) - f(y)| < n\epsilon = n \) is a contradiction.

   (c) Find \( f'(0) \).
   \[ \text{ANS: } 0. \]

3. Show that if \( f \) is differentiable in \( \mathbb{R} \) with \( f(0) = 0 \) and \( f'(0) = 1 \) then \( \exists \delta > 0 \forall 0 < x < \delta \) there is \( f(x) > 0 \).
   \[ \text{ANS: Since } \lim_{x \to 0} \frac{f(x)}{x} = 1 \text{ there is } \delta > 0 \text{ so that if } 0 < x < \delta \text{ then } \frac{f(x)}{x} > \frac{1}{2} \text{ so } f(x) > \frac{x}{2} > 0. \]

4. If \( f \in C^2\mathbb{R}, f(0) = 0, f(2) = -4 \) and \( f'(3) = 1 \) find

   (a) the largest interval which must be contained in the image of \( f' \).
   \[ \text{ANS: By the MVT there is } 0 < c < 2 \text{ with } f'(c) = -2 \text{ so by Darboux's theorem } [-2, 1] \text{ is contained in the image of } f' \text{ and a function which is linear in } [0, 2] \text{ and in } [2, 5, 3] \text{ and has nonnegative second derivative shows that this is the biggest.} \]
(b) the smallest interval which must intersect the image of $f''$.  
\textbf{ANS:} By the MVT applied to $f'$ there is $f''(e) = \frac{1 - 2}{3 - 3} = 1$, so there is some $f''(e) \in (1, 3)$. The question should have asked for some interval. Determining whether this is smallest is too much work.

5. Consider the sequence of functions $f_n(x) = (\sin(x))^n$.

(a) Find the pointwise limit.  
\textbf{ANS:} The limit is undefined for $x = (2n - \frac{1}{2})\pi$, one for $x = (2n + \frac{1}{2})\pi$ and 0 otherwise.

(b) For which intervals $[a, b]$ is the convergence uniform?  
\textbf{ANS:} Any closed interval on which the limit is 0.

6. For the power series 
\[ \sum_{n=0}^{\infty} n^2 2^n x^{2n} \]

find:

(a) the radius of convergence  
\textbf{ANS:} $2^{-\frac{1}{2}}$

(b) the second derivative at 0  
\textbf{ANS:} 4