Math 127B Spring 2020 Final Exam (48 hour take home)

Please submit via Canvas by midnght Wednesday June 10.

1. (16 points) (Derivative) Consider the two functions with $f(0) = g(0) = 0$ and otherwise $f(x) = \sin(x)[\sin(\frac{1}{x}) - 1]$ and $g(x) = [\cos(x) - 1]\cos(\frac{1}{x})$. One is differentiable at 0. The other is not.
   (a) Use the definition of the derivative to determine which is differentiable and find its derivative at 0.
   (b) Show that the other is not differentiable at 0.
2. (14 points) (MVT) Show that if $f : [0, 1] \to \mathbb{R}$ is the restriction to $[0, 1]$ of a smooth function on $\mathbb{R}$ and $\forall x \in (0, 1)$ we have $f'(x) \neq f''(x)$ then there is at most one value of $x \in [0, 1]$ at which $f(x) = f'(x)$.
3. (14 points) (uniform) Construct an example of a sequence (f_n) of functions continuous on $[-1, 1]$ and differentiable on $(-1, 1)$ which converge uniformly to f so that f is differentiable in $(-1, 0)$ and $(0, 1)$ but f' is unbounded in $(0, \frac{1}{2})$.
4. (14 points) (Taylor) Consider the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n n + 1}.$$

   (a) Find the radius of convergence for f.
   (b) Find $P_3(x)$ the degree three Taylor polynomial for the cube $f^3(x)$.
5. (14 points) (Integral) Find a partition P of $[0, 1]$ for which $U(f, P) - L(f, P) < \frac{1}{1000}$ if $f(x) = e^x$.
6. (14 points) (Improper)
   (a) Show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges at 1 then integral $\int_{x=0}^{1} f(x)dx$ exists.
   (b) Find an example of an $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R = 1$ for which the improper integral $\int_0^1 f(x)dx$ does not exist.
7. (14 points) (Fundamental) Find $f \in C^0(-1, 1)$ so that if $F(x) = \int_0^x f(t)dt$ then $F \in C^{100}(-1, 1)$ but $F \notin C^{101}(-1, 1)$.