Math 127B Spring 2020 Practice Final Answers

1. Consider the functions $f_n(x) = \sum_{k=1}^{n} \frac{e^{-kx}}{\sqrt{k}}$.

   (a) Show that f_n converges pointwise in $(0, \infty)$ but not at 0 and call the limit $f(x)$.
   ANS: Pointwise convergence for $x > 0$ follows from the ratio test: $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \sqrt{\frac{k}{k+1}} e^{-x} = e^{-x}$.
   For $x = 0$ divergence follows from the integral test.

   (b) Determine whether the convergence is uniform.
   ANS: Since each summand is continuous in $[0, 1]$ if the covergence were uniform in $(0, 1]$ the continuous partial sums would be uniformly Cauchy in $(0, 1]$ and hence in $[0, 1]$ so the pointwise limit would exist at 0.

   (c) Show that f' exists in $(0, \infty)$.
   ANS: The derivatives of the partial sums converge by the ratio test uniformly in any closed interval contained in $(0, \infty)$. Thus $f'(x)$ exists if x is in the interior of any such interval and the union of these interiors is $(0, \infty)$.

   (d) Show that $\int_{x=0}^{\infty} f(x)dx$ exists.
   ANS: Since the sum converges uniformly in any $[a, b]$, $\int_{x=a}^{b} f(x)dx = \sum k^{\frac{-3}{2}} [e^{ka} - e^{kb}] \leq \sum k^{\frac{-3}{2}}$ which converges by the integral test and $\lim_{a\to 0^+} \lim_{b\to\infty} \int_{x=a}^{b} f(x)dx$ exists since both limits are increasing and bounded.

2. Show that the vector space of real analytic functions f in $\mathbb{R}$ which are equal to their own tenth derivative ($f^{(10)} = f$) is (exactly) 10 dimensional.
   ANS: Such a real analytic function is determined uniquely by its power series $f = \sum a_k x^k = f^{(10)} = \sum \frac{(k+10)!}{k!} a_{k+10} x^k$ so a_{k+10} is determined by a_k and $\{f_n | 0 \leq n \leq 9\}$ with $f_n(x) = \sum_r \frac{x^{10r+n}}{(10r+n)!}$ form a vector space basis.

3. Show that if $S \subset [0, 1]$ is countable then there is f which is Riemann integrable in $[0, 1]$ with $\int_{x=0}^{1} f(x)dx = 0$ and $(\forall s \in S)$ we have $f(s) > 0$.
   ANS: Enumerate $S = \{s_r\}$. Set $f(s_r) = 2^{-r}$ and if $x \notin S$ then $f(x) = 0$. Write $P_n = \{\frac{k}{n} | 0 \leq k \leq n$ and note that $L(f, P_n) = 0$ and $U(f, P_n) \leq \sum_k 2 \cdot 2^{-k} n^{-1} = \frac{2}{n}$ since each s_r is in at most two intervals.

4. Find bounded functions f and g on $[-1, 1]$ so that $(fg)'''(0)$ does not exist but for every bounded function k on $[-1, 1]$ we have $(fk)''(0)$ exists.
   ANS: One example is $f(x) = x^3$ and $g(x) = \sin \frac{1}{x}$.

5. If $\{f_n\}$ is a sequence of functions converging pointwise to f define $T(\{f_n\}) = F$ with $F(x) = \lim_{n\to\infty} n(f_n(x) - f(x))$ if it exists.

Show that if $T(\{f_n\})$ and $T(\{g_n\})$ exist then $T(\{f_ng_n\})$ exists.
ANS: If $(f_n) \to f$, $(g_n) \to g$, $T(\{f_n\}) = F$ and $T(\{g_n\}) = G$ then $(f_ng_n) \to fg$, $n(f_n(x)g(x) - f(x)g(x)) \to F(x)g(x)$, $n(f_n(x)g_n(x) - f_n(x)g(x)) \to f(x)G(x)$ and $K(x) = \lim_{n\to\infty} n[f_n(x)g_n(x) - f(x)g(x)] = \lim_{n\to\infty} n[f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)] = f(x)G(x) + F(x)g(x)$ exists so $T(\{f_ng_n\}) = K$.

6. Show that if f and g are differentiable in $\mathbb{R}$ and periodic with $f(x+1) = f(x)$ and $g(x+1) = g(x)$ and $|f'| + |g'| \geq 1$ then for every $r \in \mathbb{R}$ there is c with $\frac{f'(c)}{g'(c)} = r$.
ANS: Given $r \in \mathbb{R}$ consider $h = f - rg$ with $h(0) = h(1)$ differentiable in $(0,1)$ and continuous in $[0,1]$ and choose $c \in (0,1)$ with $h'(c) = 0$ so $f'(c) = rg'(c) \neq 0$ since $|f'(c)| + |g'(c)| \neq 0$.

7. Show that if $f : (0,1] \to \mathbb{R}$ is monotone decreasing then

$$\sum_{n=1}^{\infty} \frac{1}{n^2} f\left(\frac{1}{n}\right)$$

converges iff the improper integral

$$\int_{x=0}^{1} f(x)dx$$

converges.

Hint: Consider also the two series $\sum \frac{1}{n^2 \pm n} f(\frac{1}{n})$ and their difference.
ANS: Consider the partitions $P_N = \{\frac{1}{n} | N \geq n \geq 1\}$ and note that $\int_{x=\frac{1}{n}}^{1} f(x)dx \geq L(f, P_N) = \sum_1^{N-1} [\frac{1}{n} - \frac{1}{n+1}] f(\frac{1}{n}) = \sum_1^{N-1} \frac{1}{n^2+n} f(\frac{1}{n}) \geq \sum_1^{N-1} \frac{1}{n^2} f(\frac{1}{n})$ while $\int_{x=\frac{1}{n}}^{1} f(x)dx \leq U(f, P_N) = \sum_2^N [\frac{1}{n-1} - \frac{1}{n}] f(\frac{1}{n}) = \sum_2^N [\frac{1}{n^2-n}] f(\frac{1}{n}) \leq \sum_2^N \frac{1}{n^2} f(\frac{1}{n})$.
Since both the partial sums and the integrals are increasing each converges iff it is bounded. Hence if the integral converges so does the sum and if the sum converges so does the integral.