Math 127B Spring 2020 Final Exam (48 hour take home)

Please submit via Canvas by midnght Wednesday June 10.

1. (16 points) (Derivative) Consider the two functions with $f(0) = g(0) = 0$ and otherwise $f(x) = \sin(x)[\sin(\frac{1}{x}) - 1]$ and $g(x) = [\cos(x) - 1]\cos(\frac{1}{x})$. One is differentiable at 0. The other is not.

   (a) Use the definition of the derivative to determine which is differentiable and find its derivative at 0.

   ANS: $|g'(0)| = |\lim_{h\to 0} \frac{[\cos(h)-1]\cos(\frac{1}{h})}{h} \leq |\lim_{h\to 0} \frac{[\cos(h)-1]}{h} = 1$ by L'Hospital's rule.

   (b) Show that the other is not differentiable at 0.

   For every integer n if $x = \frac{2}{(4n+1)\pi}$ and $y = \frac{2}{(4n+3)\pi}$ then $f(x) = 0$ and $f(y) = -2\sin(y)$ so if $f'(0)$ exists then $f'(0) = 0$ and $f'(0) = \lim_{y\to 0} \frac{-2\sin(y)}{y} = -2$ a contradiction.

2. (14 points) (MVT) Show that if $f : [0, 1] \to \mathbb{R}$ is the restriction to $[0, 1]$ of a smooth function on $\mathbb{R}$ and $\forall x \in (0, 1)$ we have $f'(x) \neq f''(x)$ then there is at most one value of $x \in [0, 1]$ at which $f(x) = f'(x)$.

   ANS: Assume for contradiction that $f(x) = f'(x)$ and $f(y) = f'(y)$ with $x \neq y$ and take $g = f - f'$. Thus $g(x) = g(y)$ and there is w between x and y with $g'(w) = f'(w) - f''(w) = 0$ contradicting the conditon on f.

3. (14 points) (uniform) Construct an example of a sequence (f_n) of functions continuous on $[-1, 1]$ and differentiable on $(-1, 1)$ which converge uniformly to f so that f is differentiable in $(-1, 0)$ and $(0, 1)$ but f' is unbounded in $(0, \frac{1}{2})$.

   ANS: $f_n(x) = (x^2 + \frac{1}{n})^{\frac{-1}{4}}$ works.

4. (14 points) (Taylor) Consider the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{2^n n + 1}.$$

   (a) Find the radius of convergence for f.

   ANS: 2

   (b) Find $P_3(x)$ the degree three Taylor polynomial for the cube $f^3(x)$.

   ANS: $f(x) = 1 + \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{25}x^3 + \ldots$ so multiplying termwise gives $P_3(x) = 1 + \frac{3}{3}x + (\frac{3}{9} + \frac{3}{9})x^2 + (\frac{3}{25} + \frac{6}{27} + \frac{1}{3})x^3$.

5. (14 points) (Integral) Find a partition P of $[0, 1]$ for which $U(f, P) - L(f, P) < \frac{1}{1000}$ if $f(x) = e^x$.

ANS: Since f is increasing if all intervals of P have the same length t then $U(f,P) - L(f,P) = (f(1) - f(0))t = (e-1)t < 2t$ so take P to have 2000 intervals of equal length.

6. (14 points) (Improper)

   (a) Show that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges at 1 then integral $\int_{x=0}^{1} f(x)dx$ exists.

   ANS: By Abel's theorem the series converges uniformly in $[0,1]$ and is hence continuous and therefor integrable on $[0,1]$.

   (b) Find an example of an $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $R = 1$ for which the improper integral $\int_0^1 f(x)dx$ does not exist.

   ANS: Take every $a_n = 1$.

7. (14 points) (Fundamental) Find $f \in C^0(-1,1)$ so that if $F(x) = \int_0^x f(t)dt$ then $F \in C^{100}(-1,1)$ but $F \notin C^{101}(-1,1)$.

   ANS: $f(x) = \sum_{n=0}^{\infty} 2^{-n} 3^{-99n} \cos(3^n x)$ works.