Math 127BA Midterm Exam February 12, 2021 9:00-9:50

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Please write and sign the following: **I pledge on my honor that I have not given or received any unauthorized assistance on this examination.**

1. (9 points Definition) Consider the function

$$f(x) = \begin{cases} [\cos(x) - 1]\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

   Only one of $f'(0)$ and $\lim_{x\to 0} f'(x)$ exists.
   Figure out which and compute its value.
   ANS: If the limit exists the two are equal so only $f'(0) = \lim_{x\to 0} \frac{[\cos(x)-1]\sin\frac{1}{x}}{x}$ exists. By the squeeze lemma and L'Hôspital's rule

$$|f'(0)| \leq \left|\lim_{x\to 0} \frac{\cos(x) - 1}{x}\right| = \left|\lim_{x\to 0} \frac{-\sin(x)}{1}\right| = 0.$$

2. (9 points L'Hôspital) Assume that $g(0) = 0$, $g'(0) = 1$, and $g''(0) = 2$. Find a number a so that

$$\lim_{x\to 0} \frac{g(ax) - 2g(x)}{x^2}$$

   exists and is nonzero.
   ANS: Since $g'(0)$ exists g is continuous at 0 and since $g(0) = 0$ L'Hôspital's rule applies so the limit is $\lim_{x\to 0} \frac{ag'(ax)-2g'(x)}{2x}$. Since $g''(0)$ exists g' is continuous at 0 and since $g'(0) = 1$ the limit might exist and L'Hôspital's rule applies only if $a = 2$ in which case the limit is $\lim_{x\to 0} \frac{4g''(2x)-2g''(x)}{2} = 2$.

3. (8 points Mean Value) Show that if f is a four times differentiable function on $\mathbb{R}$ and there are at least five roots (x values for which $f(x) = 0$) then $f'''' = f^{(4)}$ also has a root.
   ANS: By the mean value theorem f' has a root strictly between any two roots of f and hence at least 4 roots. Similarly f'' has at least 3, f''' at least 2 and f'''' at least one root.

4. (8 points Banach) Define $\|f\| = \|f'\|_{sup} + |f(0)|$ for $f \in C^1[-1, 1]$.
   Prove that if $\|f - g\| = 0$ then $f = g$.
   (Recall that $\|g\|_{sup} = \sup_{x\in[-1,1]} |g(x)|$.)
   ANS: If $\|f - g\| = 0$ then $\|f' - g'\|_{sup} = 0$ and $|f(0) - g(0)| = 0$. Hence $f' = g'$ and $f = g + C$ but $f(0) = g(0)$ so $f = g$.

For the last two problems consider the following four series of functions and explain your answers.

(a) $a_n(x) = \sum_{k=0}^{n} kx^k$

(b) $b_n(x) = \sum_{k=0}^{n} 2^{-k} \sin(3^k) x^k$

(c) $c_n(x) = \sum_{k=1}^{n} (\frac{x}{3} - 1)^k$

(d) $d_n(x) = \sum_{k=0}^{n} \sin\left(\frac{x}{2^k}\right)$.

5. (8 points) Which of the above four converge uniformly in $[-1, 1]$?
ANS: (a)No(It diverges at 1)
(b)Yes(It is a power series about 0 with radius of convergence at least 2)
(c)No(It diverges at 0)
(d)Yes(The derivatives converge uniformly everywhere and the series converges at the point 0 so the series converges uniformly in any bounded interval).

6. (8 points) Which of the above four converge pointwise to a differentiable function in $(-1, 1)$?
ANS: (a)Yes(It is a power series about 0 with radius of convergence 1)
(b)Yes(Same as for 5)
(c)No(Same as for 5)
(d)Yes(Same as for 5).