Math 127BB Midterm Exam February 12, 2021 11:00-11:50

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Please write and sign the following: **I pledge on my honor that I have not given or received any unauthorized assistance on this examination.**

1. (9 points Definition) Consider the function

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

   Only one of $f'(0)$ and $\lim_{x\to 0} f'(x)$ exists.
   Figure out which and compute its value.
   ANS: If the limit exists the two are equal so only $f'(0) = \lim_{x\to 0} \frac{x^3 \sin \frac{1}{x^2}}{x}$ exists. By the squeeze lemma

$$|f'(0)| \leq \left| \lim_{x\to 0} \frac{x^3}{x} \right| = 0.$$

2. (9 points L'Hôspital) Assume that g is twice differentiable. Find a number a so that

$$g''(0) = \lim_{x\to 0} \frac{g(x) + ag(0) + g(-x)}{x^2}.$$

   ANS: Since g is differentiable it is continuous so the limit could only exist if $a = -2$. In this case L'Hôspital's rule applies twice so the limit is $\lim_{x\to 0} \frac{g'(x)-g'(-x)}{2x} = \lim_{x\to 0} \frac{g''(x)+g''(-x)}{2} = g''(0)$.

3. (8 points Mean Value) Show that if f and g are four times differentiable functions on $\mathbb{R}$ and there are at least five x values for which $f(x) = g(x)$ then there is some y with $f''''(y) = g''''(y)$.
   ANS: Take $h = f - g$. By the mean value theorem h' has a root strictly between any two roots of h and hence at least 4 roots. Similarly h'' has at least 3, h''' at least 2 and h'''' at least one root y at which $f''''(y) = g''''(y)$.

4. (8 points Banach) Define $\|f\| = \|f'\|_{sup} + |f(0)|$ for $f \in C^1[-1, 1]$.
   Prove that if $\|f\| = 0$ then f is the 0 function.
   (Recall that $\|g\|_{sup} = \sup_{x\in[-1,1]} |g(x)|$.)
   ANS: If $\|f\| = 0$ then $\|f'\|_{sup} = 0$ and $|f(0)| = 0$. Hence $f' = 0$ and $f = C$ but $f(0) = 0$ so f is the 0 function.

For the last two problems consider the following four series of functions.

(a) $a_n(x) = \sum_{k=0}^{n} x^k$
(b) $b_n(x) = \sum_{k=0}^{n} 2^{-k} \cos(3^k) x^k$
(c) $c_n(x) = \sum_{k=1}^{n} \frac{1}{k^3} (\frac{x-3}{2})^k$
(d) $d_n(x) = \sum_{k=0}^{n} \sin(\frac{x}{5^k})$.

5. (8 points) Which of the above four converge uniformly in $[-1, 1]$?
ANS: (a)No(It diverges at 1)
(b)Yes(It is a power series about 0 with radius of convergence at least 2)
(c)No(It diverges at 0)
(d)Yes(The derivatives converge uniformly everywhere and the series converges at the point 0 so the series converges uniformly in any bounded interval).

6. (8 points) Which of the above four converge pointwise to a differentiable function in $(-1, 1)$?
ANS: (a)Yes(It is a power series about 0 with radius of convergence 1)
(b)Yes(Same as for 5)
(c)No(Same as for 5)
(d)Yes(Same as for 5).