5.2.7 Let

$$g_a(x) = \begin{cases} x^a \sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbb{R}$ but such that g_a' is unbounded on $[0,1]$.

We first find $a \in \mathbb{R}$ such that g_a is differentiable on $\mathbb{R}$. To do so, observe that g_a is differentiable on $\mathbb{R} \setminus \{0\}$ for all $a \in \mathbb{R}$, because x^a (for all $a \in \mathbb{R}$), $sin(x)$ and $\frac{1}{x}$ are all differentiable on $\mathbb{R} \setminus \{0\}$. It is left to consider the case where $x = 0$. From the definition of differentiability, we know g_a is differentiable at 0 if

$$\lim_{x\to 0} \frac{g_a(x) - g_a(0)}{x} = \lim_{x\to 0} \frac{x^a \sin(\frac{1}{x})}{x} = \lim_{x\to 0} x^{a-1} \sin\left(\frac{1}{x}\right) \tag{a.1}$$

exists. Here we have $-1 \leq \sin(1/h) \leq 1$ for all $h \in \mathbb{R}$, so the above limit is bounded if and only if h^{a-1} is bounded as $h \to 0$, which is the case when $a \geq 1$. When $a = 1$, the limit in (a.1) becomes $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ which does not exists (see Example 4.2.6 from Abbott). When $a > 1$, we have

$$\left|x^{a-1} \sin\left(\frac{1}{x}\right)\right| \leq |x|^{a-1} \underbrace{\left|\sin\left(\frac{1}{x}\right)\right|}_{\leq 1} \leq |x|^{a-1} \to 0 \text{ as } x \to 0 \tag{a.2}$$

This means that g_a is differentiable on $\mathbb{R}$ if and only if $a > 1$. Apply the differentiaion rules, we get for $a \geq 1$ that

$$g_a'(x) = ax^{a-1} \sin\left(\frac{1}{x}\right) - x^{a-2} \cos\left(\frac{1}{x}\right) \quad \text{for } x \neq 0 \quad \text{and} \quad g_a'(0) = \lim_{x\to 0} x^{a-1} \sin\left(\frac{1}{x}\right) = 0 \tag{a.3}$$

Now, we find $a \in \mathbb{R}_{>1}$ for which g_a' is unbounded on $[0,1]$. Note that the first term of g_a' is bounded on $[0,1]$ for all $a > 1$ because

$$\left|x^{a-1} \sin\left(\frac{1}{x}\right)\right| \leq |x|^{a-1} \leq 1^{a-1} \leq 1 < \infty$$

Therefore, it suffices to find $a > 1$ such that $x^{a-2}\cos(\frac{1}{x})$ is unbounded on $[0,1]$. Because $\cos(x)$ is also a bounded function, the problem is reduced to the case of finding $a > 1$ such that x^{a-2} is unbounded on $[0,1]$. To do so, we first show that x^{a-2} is bounded on $(0,1]$ for all $a > 1$, then consider the case where $x = 0$. Fix $x \in (0,1]$ and $a \in \mathbb{R}_{>1}$, then there exists $N \in \mathbb{N}$ such that $x > \frac{1}{N}$. It follows that for any $a > 1$ and $x \in (0,1]$, we have $|x^{a-2}| \leq |\frac{1}{x}|^{2-a} \leq N^{2-a} < \infty$. Last, we have $x^{a-2} \to \infty$ as $x \to 0^+$ if and only if $a < 2$. This means that $a \in (1,2)$.

(b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at zero.

With part (a), we first find $a > 1$ such that g_a' is continuous at 0. From (a.3) we know that g_a is differentiable on $\mathbb{R}$ for $a > 1$ with g_a' continuous on $\mathbb{R}\setminus\{0\}$. It remains to find $a > 1$ such that $\lim_{x\to 0^-} g_a'(x) = 0 = \lim_{x\to 0^-} g_a'(x)$. To do so, first recall that we showed at the end of part (a) that g_a' blows up at 0 for $a \in (1,2)$, so a has to be at least 2. Also, we know from (a.2) that the first term of g_a' in (a.3) goes to 0 as $x \to 0$ for $a > 1$. The same argument as in (a.2) shows that the second term of g_a' in (a.3) exists only when $a > 2$. Therefore, g_a' is continuous for $a \in (2, \infty)$. Now, we find $a > 2$ such that g_a' is not differentiable at 0. With the definition of differentiability, we have that g_a' is not differentiable at 0 if

$$\lim_{x\to 0} \frac{g_a'(x) - g_a'(0)}{x} = \lim_{x\to 0} \frac{ax^{a-1}\sin\left(\frac{1}{x}\right) - x^{a-2}\cos\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} \Big[\underbrace{ax^{a-2}\sin\left(\frac{1}{x}\right) - x^{a-3}\cos\left(\frac{1}{x}\right)}_{(*)}\Big]$$

does not exist. A similar argument as above shows that the limit exists for $a > 3$, so g_a is only continuously differentiable when $a \in (2,3]$.

(c) g_a is differentiable on $\mathbb{R}$ and g_a' is differentiable on $\mathbb{R}$, but such that g_a'' is not continuous at zero.

It is shown at the end of part (b) that g_a' is differentiale at 0 when $a > 3$. Because g_a is twice (actually infinitely) differentiable on $\mathbb{R} \setminus \{0\}$, we have that g_a' is differentiable on $\mathbb{R}$ when $a \in (3, \infty)$. A little computation then yields

$$g_a''(x) = a(a-1)x^{a-2} \sin\left(\frac{1}{x}\right) - (2a-2)x^{a-3} \cos\left(\frac{1}{x}\right) - x^{a-4} \sin\left(\frac{1}{x}\right)$$

Taking the limit as $x \to 0$, the first two terms become 0 for $a > 3$ and the last term does not exists for $a \leq 4$ (and is 0 for $a > 4$). We conclude that $a \in (3, 4]$ is what we want.
Note: in the above solution, instead of finding a particular value for a, we find all possible values of a that satisfy the conditions. However, a particular a with a reasonable justification are suffice for this problem.