MAT 127B
HW 2 Solutions

Exercise 1

Exactly one of the following requests is impossible. Decide which it is, and provide examples for the other three. In each case, let's assume the functions are defined on all of $\mathbb{R}$

a) Functions f and g not differentiable at zero but where fg is differentiable at zero.

b) A function f not differentiable at zero and a function g differentiable at zero where fg is differentiable at zero.

c) A function f not differentiable at zero and a function g differentiable at zero where $f+g$ is differentiable at zero.

d) A function f differentiable at zero but not differentiable at any other point.

Proof.
The request made in c) is impossible.

Examples:
All functions are defined on $\mathbb{R}$

a) Let us consider the functions

$$f(x) = |x| + 1; \qquad g(x) = \frac{1}{|x|+1}; \qquad fg(x) = 1.$$

We know $(fg)(x)$ is differentiable at 0.

We will prove $f(x), g(x)$ are not differentiable at 0.

Left Hand Derivative for f(x):

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{-h + 1 - 1}{h} = -1.$$

Right Hand Derivative for f(x):

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h + 1 - 1}{h} = 1.$$

Hence $LHD \neq RHD$, hence $f(x)$ is not differentiable at 0.

Left Hand Derivative for g(x):

$$\lim_{h \to 0^-} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{\frac{1}{-h+1} - 1}{h} = \lim_{h \to 0} \frac{1 + h - 1}{(h)(-h + 1)} = +1.$$

Right Hand Derivative for g(x):

$$\lim_{h \to 0^+} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{\frac{1}{h+1} - 1}{h} = \lim_{h \to 0} \frac{1 - h - 1}{h(h + 1)} = -1.$$

Hence $LHD \neq RHD$, hence $g(x)$ is not differentiable at 0.

Hence the request is satisfied since $f(x)$ is not differentiable at 0, $g(x)$ is not differentiable at 0 but $fg(x)$ is differentiable at 0.

b) Let the functions be:

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$g(x) = x^2,$$

$$(fg)(x) = \begin{cases} x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Hence

$$(fg)(x) = x.$$

Derivative for f(x) at $x = 0$:

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{1}{h} - 0}{h} = \lim_{h \to 0} \frac{1}{h^2} = \text{ does not exist.}$$

Since the derivative does not exist, hence $f(x)$ is not differentiable at 0.

It is clear both the functions $g(x) = x^2$ and $(fg)(x) = x$ is differentiable at 0.

c) The request is impossible.

Let us denote $h = f + g$.

We know that h is differentiable at 0, and g is differentiable at 0.

Claim 1:

If g is differentiable at 0, then $-g$ is differentiable at 0.

Proof of Claim 1.

$$(-g)'(0) = \lim_{h \to 0} \frac{-g(0+h) - (-g(0))}{h} = \lim_{h \to 0} \frac{g(0) - g(h)}{h} = -(g'(0))$$

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Since, $-g$ is differentiable at 0 and h is differentiable at 0, hence $h+(-g)$ is differentiable at 0.

$h - g = f$ is differentiable at 0.

We proved if $f+g$ is differentiable at 0, and g differentiable at 0 then f is differentiable at 0.

The contrapositive statement is:

If f is not differentiable at 0, then g or $f+g$ is not differentiable at 0 which is contradictory to the request made.

d) Consider,

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ x^2, & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

f is differentiable at 0.

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}.$$

If h is rational, $\frac{f(h)}{h} = 0$;

If h is irrational, $\frac{f(h)}{h} = h$.

Let $\{h_n\}$ be any sequence such that $h_n \longrightarrow 0$ and $h_n \neq 0$.

Consider

$$\left\{\frac{f(h_n)}{h_n}\right\}.$$

Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|h_n| < \epsilon$.

Hence for all $n > N$,

$$\left|\frac{f(h_n)}{h_n}\right| = 0 < \epsilon \text{ or } |h_n| < \epsilon \text{ if } h_n \text{ is rational or irrational respectively}$$

This implies that $\frac{f(h_n)}{h_n} \longrightarrow 0$.

Since this is true for any non-trivial sequence h_n converging to 0, hence $f'(0) = 0$ and $f(x)$ is differentiable at 0.

Let $x \neq 0$

Let $x \in \mathbb{Q}$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x+h)}{h}$$

Consider the sequence:

$\{h_n = \frac{1}{n}\}; h_n \longrightarrow 0.$

Since $x \in \mathbb{Q}$, hence $x + h_n$ is also rational.

$$\frac{f(x+h_n)}{h_n} = 0 \forall n \implies \lim_{n\to\infty} \frac{f(x+h_n)}{h_n} = 0.$$

Consider the sequence:

$\{h_n = \frac{\sqrt{2}}{n}\}; h_n \longrightarrow 0.$

Since $x \in \mathbb{Q}$, hence $x + h_n$ is irrational.

$$\frac{f(x+h_n)}{h_n} = \frac{(x+h_n)^2}{h_n} = \frac{x^2}{h_n} + 2x + h_n \;\; \forall n \implies \lim_{n\to\infty} \frac{f(x+h_n)}{h_n} = \infty.$$

This imlies that the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

does not exist.

Let $x \in \mathbb{R} \backslash \mathbb{Q}$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x+h)-x^2}{h}$$

Since $x\in\mathbb{R}$ and since $\mathbb{Q}$ is dense in $\mathbb{R}$ hence there exists a sequence of rationals converging to x, let the sequence be $r_n\longrightarrow x$.

Consider $\{h_n=r_n-x\}$.

Hence

$x+h_n=r_n\in\mathbb{Q}$ hence

$$\frac{f(x+h_n)-x^2}{h_n}=\frac{-x^2}{h_n},\implies \lim_{n\to\infty}\frac{-x^2}{h_n}=undefined.$$

Hence

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

does not exist.

Hence f is differentiable only at one point $x=0$.

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Exercise 2

Let g be defined on an interval A, and let $c\in A$.

a) Explain why $g'(c)$ in Definition 5.2.1 could have been given by:

$$g'(c)=\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}$$

b) Assume A is open. If g is differentiable at $c\in A$, show,

$$g'(c)=\lim_{h\to 0}\frac{g(c+h)-g(c-h)}{2h}$$

Proof.

a) Definition 5.2.1:

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

Let $h = x - c$.

Then $x = c + h$.

Substituting in the above defintion we have:

$$g'(c) = \lim_{c+h \to c} \frac{g(c+h) - g(c)}{c+h-c} = \lim_{c+h \to c} \frac{g(c+h) - g(c)}{h}.$$

$$x \longrightarrow c \implies x - c \longrightarrow 0 \implies h \longrightarrow 0.$$

Hence, we have

$$\lim_{c+h \to c} \frac{g(c+h) - g(c)}{h}$$

b) A is an open interval.

Then $c \in A$ implies there exists a $\delta > 0$ such that

$$(c - \delta, c + \delta) \subset A.$$

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

If $|h| < \delta$ then $h \in A$ and can be positive or negative. Moreover $-h \longrightarrow 0 \iff h \longrightarrow 0$.

Hence substituting h with $-h$ gives us,

Hence

$$f'(c) = \lim_{-h \to 0} \frac{f(c-h) - f(c)}{-h} = \lim_{h \to 0} \frac{f(c) - f(c-h)}{h}.$$

Adding the two limits, we have

$$2f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c) + f(c) - f(c-h)}{h}$$

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h}$$

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