5.2.3

(a) Use Definition 5.2.1 to produce the proper formula for the derivative of $h(x) = 1/x$.

With Def. 5.2.1, we compute for any $c \in \mathrm{Dom}(h) = \mathbb{R} \setminus \{0\}$ that

$$\lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \lim_{x \to c} \frac{1/x - 1/c}{x - c} = \lim_{x \to c} -\frac{1}{xc} = -\frac{1}{c^2}$$

which exists for all $c \in \mathrm{Dom}(h)$. Therefore, we obtain $h' : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $h'(x) = -\frac{1}{x^2}$.

(b) Combine the result in part (a) with the Chain rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.

Consider $f, g : A \to \mathbb{R}$ for an interval $A \subseteq \mathbb{R}$ with f, g differentiable at $c \in A$ and $g(c) \neq 0$. Consider the function h as in part (a). We then have

$$\begin{aligned}
\left(\frac{f}{g}\right)'(c) &= [f \cdot (h \circ g)]'(c) \\
&= f'(c) \cdot (h \circ g)(c) + f(c) \cdot (h \circ g)'(c) \\
&= f'(c) \cdot \frac{1}{g(c)} + f(c) \cdot h'(g(c)) \cdot g'(c) \\
&= f'(c) \cdot \frac{1}{g(c)} + f(c) \cdot \frac{1}{[g(c)]^2} \cdot g'(c) \\
&= \frac{f'(c)g(c) + f(c)g'(c)}{[g(c)]^2}
\end{aligned}$$

where the second equality follows from Theorem 5.2.4 (iii), the third equality from the Chain rule and the fourth equality from part (a).

(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of Theorem 5.2.3 (iii).

With the same assumption as in Theorem 5.2.4 (iv) (or part (b)), we know f and g are differentiable (hence continuous) as c. Rewriting the difference quotient yields

$$\begin{aligned}
\frac{(f/g)(x) - (f/g)(c)}{x - c} &= \frac{f(x)\frac{1}{g(x)} - f(c)\frac{1}{g(c)}}{x - c} \\
&= \frac{\left[f(x)\frac{1}{g(x)} - f(c)\frac{1}{g(x)}\right] + \left[f(c)\frac{1}{g(x)} - f(c)\frac{1}{g(c)}\right]}{x - c} \\
&= \left[\frac{f(x) - f(c)}{x - c}\right]\frac{1}{g(x)} + f(c)\left[\frac{1/g(x) - 1/g(c)}{x - c}\right]
\end{aligned}$$

Because f is continuous at c, as $x \to c$, we obtain

$$\lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c}\right]\frac{1}{g(x)} = \frac{1}{g(x)} \cdot \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \frac{f'(c)}{g(x)} = \frac{f'(c)g(x)}{[g(x)]^2}$$

Similarly, becasue g is continous as c, we get

$$\lim_{x \to c} f(c)\left[\frac{1/g(x) - 1/g(c)}{x - c}\right] = -f(c) \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \cdot \lim_{x \to c} \frac{1}{g(x)g(c)} = \frac{-f(c)g'(c)}{[g(c)]^2}$$

Putting these together yields the result.