MAT 127B
HW 4 Solutions(5.2.5/5.2.10)

Exercise 1 (5.2.5)

Let

$$f_a(x) = \begin{cases} x^a, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

(a) For which values of a is f continuous at zero?

(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

(c) For which values of a is f twice-differentiable?

Proof.
The function $f_a(x)$ is smooth for every a for all points in $\mathbb{R} - \{0\}$.

(a) For continuity we need

$$\lim_{x \to 0^+} f_a(x) = f_a(0) = \lim_{x \to 0^-} f_a(x)$$

Hence,

$$f_a(0) = \lim_{x \to 0^-} f_a(x) = 0$$

$$\lim_{x \to 0^+} f_a(x) = \lim_{x \to 0^+} x^a = 0$$

if and only if $a > 0$

(b) For differentiability we need

$$\lim_{h\to 0^+} \frac{f_a(h)-f_a(0)}{h} = \lim_{h\to 0^-} \frac{f_a(h)-f_a(0)}{h}$$

Hence,

$$\lim_{h\to 0^-} \frac{f_a(h)-f_a(0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

We need

$$\lim_{h\to 0^+} \frac{f_a(h)-f_a(0)}{h} = \lim_{h\to 0} \frac{h^a - 0}{h} = \lim_{h\to 0} h^{a-1} = 0.$$

This is true if and only if $a-1>0$ or $a>1$. and $f_a'(0)=0$.

The derivative will be (when $a>1$):

$$f_a'(x) = \begin{cases} ax^{a-1}, & \text{if } x>0 \\ 0, & \text{if } x\leq 0 \end{cases}$$

$$\begin{aligned} \lim_{x\to 0^+} f_a'(x) &= \lim_{x\to 0^+} ax^{a-1} = 0 \\ &= f_a'(0) = \lim_{x\to 0^-} f_a'(x) = 0 \end{aligned}$$

Hence the derivative of $f_a(x)$ is continuous for $a>1$.

(c) For twice differentiability we need

$$\lim_{h\to 0^+} \frac{f_a'(h)-f_a'(0)}{h} = \lim_{h\to 0^-} \frac{f_a'(h)-f_a'(0)}{h}$$

Hence,

$$\lim_{h\to 0^-} \frac{f_a'(h)-f_a'(0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

We need

$$\lim_{h\to 0^+} \frac{f_a'(h) - f_a'(0)}{h} = \lim_{h\to 0} \frac{ah^{a-1} - 0}{h} = \lim_{h\to 0} ah^{a-2} = 0.$$

This is true if and only if $a - 2 > 0$ or $a > 2$. and $f_a''(0) = 0$.

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Exercise 2(5.2.10)

Recall that a function $f : (a, b) \longrightarrow \mathbb{R}$ is increasing on (a, b) if $f(x) \leq f(y)$ whenever $x < y$ in (a, b). A familiar mantra from calculus is that a differentiable function is increasing if its derivative is positive, but this statement requires some sharpening in order to be completely accurate.
Show that the function

$$g(x) = \begin{cases} \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is differentiable on $\mathbb{R}$ and satisfies $g'(0) > 0$.
Now, prove that g is not increasing over any open interval containing 0.
In the next section we will see that f is indeed increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.

Proof.
For $x \neq 0$, we have $g(x)$ is differentiable and the derivative is:

$$g'(x) = \frac{1}{2} + 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) = \frac{1}{2} + 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

For $x = 0$

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{\frac{h}{2} + h^2\sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0}\left(\frac{1}{2} + h\sin\left(\frac{1}{h}\right)\right) = \frac{1}{2} > 0$$

since

$$\lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$$

Let $(-a, b); a, b > 0$ be any interval containing 0.

Consider the sequence:

$$s_n = \frac{2}{(4n-1)\pi}; n \in \mathbb{N}.$$

Clearly this sequence converges to 0.
Hence there exists an N such that for all $n > N$, $s_n < b$. It's clear that $s_n > 0$, hence for all $n > N$,
$s_n \in (-a, b)$.
Let $n_0 = N + 1$.
Now consider

$$0 < x = \frac{2}{(4n_0+1)\pi} < \frac{2}{(4n_0-1)\pi} = y < b.$$

We have $x < y$. We will show $g(x) > g(y)$ which implies g is not increasing.

$g(x) > g(y)$ is equivalent to showing $g(x) - g(y) > 0$.

$$\begin{aligned} g(x) &= \frac{2}{(4n_0+1)(\pi)(2)} + \frac{4}{(4n_0+1)^2(\pi)^2} \sin\left(\frac{(4n_0+1)\pi}{2}\right) \\ &= \frac{1}{(4n_0+1)(\pi)} + \frac{4}{(4n_0+1)^2(\pi)^2} \sin\left(\frac{(4n_0+1)\pi}{2}\right) \\ &= \frac{1}{(4n_0+1)(\pi)} + \frac{4}{(4n_0+1)^2(\pi)^2} \end{aligned}$$

$$g(y) = \frac{1}{(4n_0-1)(\pi)} - \frac{4}{(4n_0-1)^2(\pi)^2}$$

$$\begin{aligned} g(x) - g(y) &= \frac{1}{(4n_0+1)(\pi)} + \frac{4}{(4n_0+1)^2(\pi)^2} - \frac{1}{(4n_0-1)(\pi)} + \frac{4}{(4n_0-1)^2(\pi)^2} \\ &= \left(\frac{1}{(4n_0+1)(\pi)} - \frac{1}{(4n_0-1)(\pi)}\right) + \left(\frac{4}{(4n_0+1)^2(\pi)^2} + \frac{4}{(4n_0-1)^2(\pi)^2}\right) \\ &= \left(\frac{-2}{(16n_0^2-1)(\pi)}\right) + \left(\frac{(8)(16n_0^2+1)}{(16n_0^2-1)^2(\pi)^2}\right) \end{aligned}$$

$$= \frac{(-2)(16n_0^2-1)(\pi)+8(16n_0^2+1)}{(16n_0^2-1)^2(\pi)^2} = \frac{16n_0^2(8-2\pi)+(2\pi+8)}{(16n_0^2-1)^2(\pi)^2} > 0$$

since $8-2\pi > 0$ and $2\pi+8>0$ and every other term is a square.

Hence $x<y$ but $g(x)>g(y)$.

Since $(-a,b)$ was an arbitrary open interval of 0, hence g is not increasing in any open interval of 0.

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