MAT 127B
HW 6 Solutions(5.3.9/5.3.10)

Theorem 5.3.6

Let f and g be continuous on an interval containing a, and assume f and g are differentiable on this interval with the possible exception of the point a. If $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for all $x \neq a$, then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \textit{ implies } \lim_{x\to a} \frac{f(x)}{g(x)} = L$$

Exercise 1(5.3.9)

Assume f and g are as described in Theorem 5.3.6, but now add the assumption that f and g are differentiable at a, and f' and g' are continuous at a with $g'(a) \neq 0$. Find a short proof for the 0/0 case of L'Hospital's Rule under this stronger hypothesis.

Proof.
We are given $g'(a) \neq 0$.
Since g' is continuous at 0 implies there exists $h \in \mathbb{R}$ such that, $g'(x) \neq 0$ for all $x \in (a - h, a + h)$.

This is easy to prove. Let for each n, consider the open set $(a - \frac{1}{n}, a + \frac{1}{n})$.
If for each of these open sets, there exists a point a_n such that $g'(a_n) = 0$, then we have the sequence $a_n \longrightarrow a$ and since g' is continuous, hence $g'(a_n) \longrightarrow g'(a)$.

Since $g'(a_n) = 0 \forall n$, it forces $g'(a) = 0$ which is not true. Hence there exists an $n_0 \in \mathbb{N}$ such that $g'(x) \neq 0$ for all $x \in (a - \frac{1}{n_0}, a + \frac{1}{n_0})$. We call $h = \frac{1}{n_0}$

Now if we consider the interval $(a-h/2, a+h/2)$, for every $x \neq a$ in the interval, if we apply the mean value theorem to f, we get:

$$|g(x) - g(a)| = |g'(\theta)||x - a|.$$

where θ is a number between x and a
If $g(x) = 0$ for some $x \neq a$, $x \in (a - h/2, a + h/2)$, then

$$0 = |g'(\theta)||x - a|$$

which would imply $g'(\theta) = 0$ for $\theta \in (a - h/2, a + h/2)$ which is contradictory.

Hence $g(x) \neq 0$ for $x \in (a - h/2, a + h/2)$ and $x \neq a$. We will call $(a - h/2, a + h/2) = I$
If,

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L,$$

then since f and g are continuous functions and $g'(a) \neq 0$. hence,

$$L = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a; x\in I} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$
$$= \lim_{x\to a; x\in I} \frac{(f(x) - f(a))(x - a)}{(g(x) - g(a))(x - a)} = \lim_{x\to a; x\in I} \frac{f(x)}{g(x)}.$$

The last term is defined when $x \in I$ since $g(x) \neq 0$ for all $x \in I \backslash \{a\}$.

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Exercise 2(5.3.10)

Let

$$f(x) = x \sin\left(\frac{1}{x^4}\right) e^{-\frac{1}{x^2}}$$

and

$$g(x) = e^{-\frac{1}{x^2}}.$$

Using the familiar properties of these functions, compute the limit as x approaches zero of $f(x)$, $g(x)$, $f(x)/g(x)$, and $f'(x)/g'(x)$. Explain why the results are surprising but not in conflict with the content of Theorem 5.3.6.

Proof.

$$\lim_{x\to 0} f(x) = \left(\lim_{x\to 0} x \sin\left(\frac{1}{x^4}\right)\right)\left(\lim_{x\to 0} e^{-\frac{1}{x^2}}\right) = 0.$$

Hence define $f(0) = 0$.

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} e^{-\frac{1}{x^2}} = 0.$$

Hence define $g(0) = 0$.

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} x \sin\left(\frac{1}{x^4}\right) = 0.$$

$$f'(x) = \sin\left(\frac{1}{x^4}\right) e^{-\frac{1}{x^2}} + x\cos\left(\frac{1}{x^4}\right)\left(\frac{-4}{x^5}\right) e^{-\frac{1}{x^2}} + x\sin\left(\frac{1}{x^4}\right) e^{-\frac{1}{x^2}} \frac{2}{x^3}.$$

$$g'(x) = e^{-\frac{1}{x^2}}\left(\frac{2}{x^3}\right).$$

$$\begin{aligned}\lim_{x\to 0} \frac{f'(x)}{g'(x)} &= \lim_{x\to 0} \sin\left(\frac{1}{x^4}\right)\frac{x^3}{2} + x\cos\left(\frac{1}{x^4}\right)\left(\frac{-4}{x^5}\right)\frac{x^3}{2} + x\sin\left(\frac{1}{x^4}\right). \\ &= \lim_{x\to 0} \frac{-2}{x} \cos\left(\frac{1}{x^4}\right) = \text{ unbounded.}\end{aligned}$$

The result is surprising since it satisfies the conditions of f and g in a small neighbourhood of $x = 0$, in the theorem but

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} \neq \lim_{x\to 0} \frac{f(x)}{g(x)}.$$

Though the result is surprising but it is not in conflict with the Theorem since the Theorem holds when

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = L < \infty$$

which implies

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f(x)}{g(x)}$$

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