[Darboux Problems] Write $G_a(x) = x^a \sin(\frac{1}{x})$ if $x \neq 0$ and $G_a(0) = 0$. Similarly write $F_a(x) = x^a \cos(\frac{1}{x})$ if $x \neq 0$ and $F_a(0) = 0$

(a) Show that $f(x) = G_2'(x)$ if $x \neq 0$ and $f(0) = \frac{1}{2}$ satisfies the Intermediate Value Property (the conclusion of 7.44) but is not a derivative function.

A little computation shows that $f(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$ for $x \neq 0$ and $f(0) = \frac{1}{2}$. Because f is continuous on $\mathbb{R} \setminus \{0\}$, it is continuous on any compact subset of $\mathbb{R} \setminus \{0\}$. We want to show (wlog) that for any $[c, d]$ with $f(c) < 0 < f(d)$ there exists $b \in (c, d)$ with $f(b) = 0$. To do this, we consider the two cases where $cd > 0$ and $cd \leq 0$. If $cd > 0$, then $c, d \in \mathbb{R}_-$ or $c, d \in \mathbb{R}_+$. Apply Darboux's Theorem to G_a on $[c, d]$ and we can show f satisfies the IVP on $[c, d]$. To prove the case where $cd \leq 0$, we find $c', d' \in (0, \epsilon)$ with $f(c') < 0 < f(d')$ for ϵ arbitrarily small. Then the second case is reduced to the first case and we are done. Now, to find such c' and d', we observe that $\sin(x) = 0$ when $\cos(x) = \pm 1$. This means that $f(x) = 1$ when $\cos(\frac{1}{x}) = -1$ and $f(x) = -1$ when $\cos(\frac{1}{x}) = 1$. Indeed, because $\cos(\frac{1}{x}) = -1$ for $x = \frac{1}{\pi + 2n\pi}, n \in \mathbb{N}$ and $\cos(\frac{1}{x}) = 1$ for $x = \frac{1}{2n\pi}, n \in \mathbb{N}$, we can pick $N \in \mathbb{N}$ large enough (with the Archimedean property) so that $0 < c' = \frac{1}{2(N+1)\pi} < d' = \frac{1}{\pi + 2N\pi} < \epsilon$. We just showed that f satisfies the Intermediate Value Property. To show f is not a derivative function, recall from Exercise 5.2.7(a) that $G_2'(0) = 0$. This means that $f(x) - G_2'(x)$ is a function takes on only 2 values, and therefore is not a derivative function by the consequence of the Darboux theorem. Hence, f is not a derivative function follows from the linearity of derivative. □

(b) Show that the derivative function $g(x) = G_{\frac{3}{2}}'(x)$ has $\text{Im}(g|_{[0,1]})$ unbounded.

This is a consequence of the solution of Exercise 5.2.7(a) in HW 1.

(c) Show that the derivative function $g(x) = [G_2(x) - 2x^2]'$ has $\text{Im}(g|_{[0,1]})$ not closed.

With some computation, we get $g(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}) - 4x$ for $x \neq 0$ and $g(0) = 0$. To prove the statement, we first observe that $g(x)$ is continuous on $(0, 1]$ and any compact subset of $(0, 1]$. We also recall from 127A that continuous functions map compact set to compact set on $\mathbb{R}$. This means that $\text{Im}(g|_{[\epsilon,1]})$ is closed for any $\epsilon \in (0, 1)$. Therefore, for $\text{Im}(g|_{[0,1]})$ to be not closed, the accumulation point of $\text{Im}(g|_{[0,1]})$ that is not in $\text{Im}(g|_{[0,1]})$ has to appear as $x \to 0$. Also, note that $\sin(\frac{1}{x}) = 0 \Leftrightarrow \cos(\frac{1}{x}) = \pm 1$ and $\cos(\frac{1}{x}) - 4x \to \cos(\frac{1}{x})$ from below as $x \to 0$. Putting these together, it is reasonable that we pick the sequence $\{x_n\}_{n=1}^{\infty} := \{\frac{1}{\pi + 2n\pi}\}_{n=1}^{\infty}$ for which $-cos(\frac{1}{x_n}) = 1$ and $2x\sin(\frac{1}{x_n}) = 0$ for all $n \in \mathbb{N}$. Indeed, as $n \to \infty$, we get $x_n \to 0$ and $g(x_n) = 1 - \frac{1}{\pi + 2n\pi} \to 1$. Now, we have found a sequence $g(x_n) \in \text{Im}(g|_{[0,1]})$ which converges to 1. If we can show that $1 \notin \text{Im}(g|_{[0,a]})$, then we have found our accumulation point. Indeed, we have for $x > 0$ that

$$g(x) = 2x \underbrace{\sin(\frac{1}{x})}_{\leq 1} - \cos(\frac{1}{x}) - 4x \leq -\cos(\frac{1}{x}) - 2x < -\cos(\frac{1}{x}) \leq 1,$$

The proof is complete. □

(d) Show that although $h(x) = G_2'(x)$ and $k(x) = F_2'(x)$ are derivative functions, $h^2(x)$ and $k^2(x)$ cannot both be derivative functions. (Hint: compute $h^2 + k^2$ explicitly.)

We compute $h(x)^2 + k(x)^2$ explicitly. We know $h(0)^2 + k(0)^2 = 0$. For $x \neq 0$, we have

$$\begin{aligned} h(x)^2 + k(x)^2 &= \left[2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})\right]^2 + \left[2x\cos(\frac{1}{x}) + \sin(\frac{1}{x})\right]^2 \\ &= 4x^2 \underbrace{\left[\sin(\frac{1}{x})^2 + \cos(\frac{1}{x})^2\right]}_{=1} + \underbrace{\left[\sin(\frac{1}{x})^2 + \cos(\frac{1}{x})^2\right]}_{=1} \\ &= 4x^2 + 1 \end{aligned}$$

FSC, assume $h^2(x)$ and $k^2(x)$ are both derivative functions, then $h^2(x) + k^2(x)$ is a derivative function as well. However, we know $\text{Im}(h^2 + k^2) = \{0\} \cup (1, \infty)$, which contradicts the Darboux's theorem (i.e. we cannot find x with $h^2(x) + k^2(x) = \frac{1}{2}$. □

(e) Show that there are derivative functions $m(x)$ and $n(x)$ for which the composition $(m \circ n)(x) = m(n(x))$ exists but is not a derivative function.

From part (d) we know that $h^2(x)$ and $k^2(x)$ cannot both be derivative functions. Denote the one for which its square is not a derivative function by $n(x)$. Let $m(x) = x^2$. We know $m(x)$ is the derivative function of $\frac{1}{3}x^3$ and $m(n(x))$ exists but is not a derivative function by part (d). □