**6.2.6** Assume $f_n \to f$ on a set $A$. Theorem 6.2.6 is an example of a typical type of question which asks whether a trait possessed by each $f_n$ is inherited by the limit function. Provide an example to show that *all* of the following propositions are false if the convergence is only assumed to be pointwise on $A$. Then go back and decide which are true under the stronger hypothesis of uniform convergence.

(a) If each $f_n$ is uniformly continuous, then $f$ is uniformly continuous.

Consider $f_n(x) = x^n$ on $[0,1]$. We know each $f_n$ is uniformly continuous because

$$|x^n - y^n| \le |x-y| \underbrace{\left|\sum_{i=0}^{n-1} x^i y^{(n-1)-i}\right|}_{\le n} \le n|x-y|$$

However, the pointwise limit of $f_n$ is $f(x) = 0$ for $x \in [0,1)$ and $f(x) = 1$ for $x = 1$, and $f$ is not uniform continuous because $f$ is not continuous.

Now, assume $f_n \to f$ uniformly on $A$ and $f_n$ uniformly continuous for all $n \in \mathbb{N}$. Then for any $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that for all $x \in A$, $|f_n(x) - f(x)| < \epsilon$ for $n > N_\epsilon$. Also, because $f_n$ is uniformly continuous, we know for any $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that $|x-y| < \delta_\epsilon \Rightarrow |f_n(x) - f_n(y)| < \epsilon$. Therefore, for each $\epsilon > 0$, pick $N_\epsilon$ and $\delta_\epsilon$ as above and we get for all $n > N_\epsilon$ and $|x-y| < \delta_\epsilon$ that

$$\begin{aligned}|f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< 3\epsilon\end{aligned}$$

so $f(x)$ is uniformly continuous. □

(b) If each $f_n$ is bounded, then $f$ is bounded.

Consider $f_n(x)$ as in Exercise 6.2.1 in Abbott with $A = (0,1)$. We have $f_n$ is bounded because $nx < n$ for $x \in (0,1)$ and $nx^2 > 0$. But its pointwise limit $f = \frac{1}{x}$ is unbounded on $(0,1)$.

For the proof of the uniform convergent case, see Theorem 9.14 in Hunter's notes.

(c) If each $f_n$ has finite number of discontinuities, then $f$ has a finite number of discontinuities.

The sequence of functions $g_n(x)$ in part (b) of Exercise 6.2.2 in Abbott is a counterexample for both cases.

(d) If each $f_n$ has fewer than $M$ discontinuities (where $M \in \mathbb{N}$ is fixed), then $f$ has fewer than $M$ discontinuities.

Consider the sequence of functions on $[0,1]$ with

$$f_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{M}, \frac{2}{M}, \cdots, \frac{M-1}{M} \\ x^n & \text{otherwise} \end{cases}$$

where each $f_n$ has $M-1$ discontinuities. However, we have its limit function

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{M}, \frac{2}{M}, \cdots, \frac{M-1}{M}, 1 \\ 0 & \text{otherwise} \end{cases}$$

has $M$ discontinuities.

Now, assume $f_n \to f$ uniformly. Fsc assume each $f_n$ has fewer than $M$ discontinuities but $f$ has $M$ discontinuities (other cases can be reduced to this one). We will show that infinite many $f_n$ must also have $M$ discontinuities, contradicting our assumption.

Denote the $M$ discontinuities of $f$ by $x_1, x_2, \cdots, x_M$. We then have for any $\delta > 0$ that there exist

$x_i' \in V_\delta(x_i) \setminus \{x_i\}$ and $\epsilon_i > 0$ such that $|f(x_i') - f(x_i)| \geq \epsilon_i$, for $i = 1, 2, \cdots, M$. For simplicity set $\epsilon := \min\{\epsilon_1, \cdots, \epsilon_M\}$. Because $f_n \to f$ uniformly, for $\frac{\epsilon}{4} > 0$ there exists $N_{\frac{\epsilon}{4}} \in \mathbb{N}$ such that for all $n > N_{\frac{\epsilon}{4}}$ we have $|f_n(x_i) - f(x_i)| < \frac{\epsilon}{4}$ and $|f_n(x_i') - f(x_i')| < \frac{\epsilon}{4}$ where $i = 1, \cdots, M$. Putting these together we obtain for all any $\delta > 0$, there exists $\frac{\epsilon}{4} > 0$, $N_{\frac{\epsilon}{4}} \in \mathbb{N}$ and $x_i'$ in the deleted $\delta$-nbhd of $x_i$ such that $|f_n(x_i') - f_n(x_i)| \geq \frac{\epsilon}{4}$ for $i = 1, \cdots, M$. This completes the proof. □

Note: for the uniform convergent case, simply arguing with the Continuous Limit Theorem (Thm 6.2.6) is not enough, because $f_n$'s might have discontinuities at different $x$'s.

(e) If each $f_n$ has at most a countable number of discontinuities, then $f$ has at most a countable number of discontinuities.

Consider the sequence of functions on $[0, 1]$ with

$$f_n(x) = \begin{cases} 1 & \text{if } x = \frac{i}{m} \text{ with } i = 1, \cdots, m \text{ and } m = 1, \cdots, n \\ 0 & \text{otherwise} \end{cases}$$

Then each $f_n$ has at most $\frac{n^2+n}{2}$ discontinuities. However, as $n \to \infty$, we know for each irrational number $r \in \mathbb{R} \setminus \mathbb{Q}$, there exists a sequence of rational numbers $\{x_k\} \subseteq \mathbb{Q}$ with $f_n(x_k) = 1$ for all $k \in \mathbb{N}$ and $x_k \to r$ as $k \to \infty$. This means that as $n \to \infty$, $f_n$ has discontinuities at every irrational number between $[0, 1]$, which is uncountable.

We can modify the above $f_n$ so that it converges uniformly. Consider the sequence of functions on $[0, 1]$

$$f_n(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ with } p, q \in \mathbb{N},\ \gcd(p, q) = 1, \text{ and } q \leq n \\ 0 & \text{otherwise} \end{cases}$$

Then $f_n$ converge uniformly to

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ with } p, q \in \mathbb{N},\ \gcd(p, q) = 1, \text{ and } q \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

with a similar argument as in Exercise 6.2.2(b) from HW 9.

**6.2.7** Let $f$ be uniformly continuous on all of $\mathbb{R}$, and define a sequence of functions by $f_n(x) = f(x + \frac{1}{n})$. Show that $f_n \to f$ uniformly. Give an example to show that this proposition fails if is only assumed to be continuous and not uniformly continuous on $\mathbb{R}$.

Fix $\epsilon > 0$. The uniform continuity of $f$ implies that there exists $\delta > 0$ such that for $< |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. We pick $N \in \mathbb{N}$ such that $\frac{1}{N} > \delta$. Then we have for all $n > N$ and all $x \in \mathbb{R}$ that

$$|x + \frac{1}{n} - x| = \frac{1}{n} < \frac{1}{N} < \delta \quad \Rightarrow \quad |f_n(x) - f(x)| = |f(x + \frac{1}{n}) - f(x)| < \epsilon$$

Hence, $f_n \to f$ uniformly.

To show this proposition fails if $f$ is continuous but not uniformly continuous on $\mathbb{R}$, consider $f(x) = x^2$ on $[0, 1]$. Then we have $|f_n(x) - f(x)| = |(x + \frac{1}{n})^2 - x^2| = \frac{2x}{n} + \frac{1}{n^2}$. For each $n \in \mathbb{N}$, pick $x = n$ and $\epsilon = \frac{1}{2}$ then we get $|f_n(n) - f(n)| \geq 1 > \epsilon$, so the proposition fails. □

**6.2.10** This exercise and the next explore partial converses of the Continuous Limit Theorem (Theorem 6.2.6). Assume $f_n \to f$ pointwise on $[a, b]$ and the limit function $f$ is continuous on $[a, b]$. If each $f_n$ is increasing (but not necessarily continuous), show $f_n \to f$ uniformly.

First note that the set $[a, b]$ is compact on $\mathbb{R}$. This means that every open cover of $[a, b]$ has a finite subcover. In addition, because $f$ is continuous on $[a, b]$, $f$ is uniformly continuous on $[a, b]$. Fix $\epsilon > 0$ and we can pick $\delta > 0$ such that $0 < |x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2}$. Moreover, we can show that because $f_n \to f$ to $f$ pointwise, each $f_n$ is increasing, and $f$ is continuous that $f$ must also be increasing.

Consider the finite cover of $[a, b]$ with radius $\delta$ centered around $\{x_i\}_{i=1}^k \subseteq [a, b]$ for some $k \in \mathbb{N}$. We know for the above $\epsilon > 0$ there exists $N_1, \cdots, N_k$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in [a, b]$ and all $n > N_i$, $i = 1, \cdots, N$, respectively. Choose $N := \min\{N_1, \cdots, N_k\}$. We then have for all $n > N$ and for all $x \in [a, b]$ that $x \in [x_i, x_{i+1}]$ for some $i = 1, \cdots, N - 1$. Consider the two cases $f_n(x) > f(x)$ and $f_n(x) < f(x)$, we have for the first case that

$$f(x_i) < f(x) < f_n(x) < f_n(x_{i+1}) < f(x_{i+1}) + \frac{\epsilon}{2} < f(x_i) + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

and for the second case that

$$f(x_i) - \frac{\epsilon}{2} < f_n(x_i) < f_n(x) < f(x) < f(x_{i+1}) < f(x_i) + \frac{\epsilon}{2}$$

This means that for all $n > N$ and $x \in [a, b]$,

$$|f_n(x_{i+1}) - f_n(x)| < f(x_i) - f(x_i) < \epsilon$$

so $f_n \to f$ uniformly. □