MAT 127B
HW 12 Solutions(6.3.2/6.3.4/6.3.6/6.3.7)

Exercise 1 (6.3.2)

Consider the sequence of functions

$$h_n(x) = \sqrt{x^2 + \frac{1}{n}}$$

a) Compute the pointwise limit of (h_n) and then prove that the convergence is uniform on $\mathbb{R}$.

b) Note that each h_n is differentiable. Show $g(x) = \lim h_n'(x)$ exists for all x, and explain how we can be certain that the convergence is not uniform on any neighborhood of zero.

Proof.

a)

$$\lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \sqrt{x^2 + \frac{1}{n}} = \sqrt{x^2} = |x|$$

Hence the pointwise limit of $\{h_n(x)\}$:
$h(x) = |x|$.

Given $\epsilon > 0$, we need to exibit a $N \in \mathbb{N}$ such that for all $n \geq N$ and for all $x \in \mathbb{R}$,

$$|h_n(x) - h(x)| = \left| \sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2} \right| < \epsilon.$$

Given $\epsilon > 0$,

choose $N \in \mathbb{N}$, such that

$$N > \frac{1}{\epsilon^2}.$$

Let $n \geq N$ and let $x \in \mathbb{R}$. implies

$$\sqrt{\frac{1}{n}} \leq \sqrt{\frac{1}{N}} < \epsilon.$$

Consider

$$2\sqrt{\frac{x^2}{n}} \geq 0.$$

Hence

$$x^2 + \frac{1}{n} \leq x^2 + \frac{1}{n} + 2\sqrt{\frac{x^2}{n}} = \left(\sqrt{x^2} + \sqrt{\frac{1}{n}}\right)^2$$

$$\implies \sqrt{x^2 + \frac{1}{n}} \leq \sqrt{x^2} + \sqrt{\frac{1}{n}}$$

$$\left|\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2}\right| \leq \sqrt{\frac{1}{n}} \leq \sqrt{\frac{1}{N}} < \epsilon.$$

Hence, $h_n(x)$ converges to $h(x)$ uniformly.

b) Each h_n is differentiable since

$$h_n'(x) = \lim_{h \to 0} \frac{h_n(x+h) - h_n(x)}{h} = \frac{x}{(\sqrt{x^2 + \frac{1}{n}})}$$

exists for $x \in \mathbb{R}$.

$$g(x; x \neq 0) = \lim_{n \to \infty} h_n'(x) = \lim_{n \to \infty} \frac{x}{\sqrt{x^2 + \frac{1}{n}}} = \frac{x}{|x|}$$

exists for $x \neq 0$.

For $x = 0$,

$$g(0) = \lim_{n \to \infty} h_n'(0) = \lim_{n \to \infty} \frac{0}{\sqrt{0 + \frac{1}{n}}} = 0.$$

Note that for each n, $h_n'(x)$ is continuous at $x = 0$.

Now, $\lim_{n\to\infty} h_n'(x) = g(x)$.

ALso $g(x)$ is not continuous at $x = 0$.

If $h_n'(x) \longrightarrow g(x)$ is uniform in any interval containing 0 and also given each $h_n'(x)$ is continuous at 0, implies $g(x)$ should be continuous at $x = 0$, which is not the case.

Hence the convergence is not uniform.

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Exercise 2 (6.3.4)

Let

$$h_n(x) = \frac{sin(nx)}{\sqrt{n}}.$$

Show that $h_n \longrightarrow 0$ uniformly on $\mathbb{R}$ but that the sequence of derivatives (h_n') diverges for every $x \in \mathbb{R}$.

Proof.
Given $\epsilon > 0$,
Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\epsilon^2}.$$

For any $n \geq N$, and for any $x \in \mathbb{R}$,

$$\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \epsilon.$$

Hence

$$|h_n(x) - 0| = \left|\frac{\sin(nx)}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n}} < \epsilon.$$

Hence $h_n \longrightarrow 0$ converges uniformly.

$$h_n'(x) = \sqrt{n}\cos(nx).$$

Let $x = \pi x'$.
Either x' is rational or irrational.

If $x' = \frac{p}{q}$, $q > 0$ then consider the subsequence
$n = \{2q, 4q, 6q, \ldots, 2kq, \ldots\}$.
Hence we have the subsequence

$$\{\sqrt{2kq}\cos(pk2\pi n)\} = \sqrt{2kq} \longrightarrow \infty.$$

Say x' is irrational.
Consider the set

$$\{n\pi x'(mod 2\pi)\}_n \subset [0, 2\pi].$$

Every element of this set is distinct since otherwise for $n_1 \neq n_2$:

$$n_1\pi x' - n_2\pi x' = 2k\pi \implies x' = \frac{2k}{n_1 - n_2}$$

which implies x' is rational.

We have a sequence in a compact set $[0, 2\pi]$ which implies there is a convergent subsequence. Consider the subsequence $\{n_k\pi x'(mod 2\pi)\}$ which converges. If the sequence converges to $\pi/2$ or $3\pi/2$, then consider the sequence,

$$\{2n_k\pi x'(mod 2\pi)\} \longrightarrow \pi.$$

Since $\cos x$ is a continuous function, hence say

$$\{n_k\pi x'(mod 2\pi)\} \longrightarrow y,$$

with $\cos y \neq 0$.
Choose ϵ such that,
$|\cos y| > \epsilon > 0$,
This implies there exists a $\delta > 0$, such that whenever

$$|x - y| < \delta \implies |\cos x - \cos y| < \epsilon \implies \cos y - \epsilon < \cos x < \cos y + \epsilon.$$

Given δ, there exists $N \in \mathbb{N}$, such that for all $k \geq N$,

$$|n_k\pi x'(mod 2\pi) - y| < \delta.$$

If $\cos y > 0$ (similarly for $\cos y < 0$)
then we have for all $k \geq N$

$$0 < \cos y - \epsilon < \cos(n_k\pi x') \ \ (\cos(n_k\pi x') < \cos y + \epsilon < 0)$$

Hence,
consider the subsequence for $k \geq N$,

$$\text{if } (\cos y > 0) \qquad \lim_{k\to\infty} h_{n_k} = \lim_{k\to\infty} \sqrt{n_k}\cos(n_k\pi x') > \lim_{k\to\infty} \sqrt{n_k}(\cos(y) - \epsilon) = \infty$$

OR

$$\text{if } (\cos y < 0) \qquad \lim_{k\to\infty} h_{n_k} = \lim_{k\to\infty} \sqrt{n_k}\cos(n_k\pi x') < \lim_{k\to\infty} \sqrt{n_k}(\cos(y) + \epsilon) = -\infty$$

Hence the sequence $\{h_n(x)\}$ diverges for every x.

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Exercise 3 (6.3.6)

Provide an example or explain why the request is impossible. Let's take the domain of the functions to be all of R.

a) A sequence (f_n) of nowhere differentiable functions with $f_n \longrightarrow f$ uniformly and f everywhere differentiable.

b) A sequence (f_n) of differentiable functions such that (f_n') converges uniformly but the original sequence (f_n) does not converge for any $x \in \mathbb{R}$.

c) A sequence (f_n) of differentiable functions such that both (f_n) and (f_n')converge uniformly but $f = \lim f_n$ is not differentiable at some point.

Proof.

a) Let

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x),$$

where $h(x) = |x|$ on $[-1, 1]$ and extended to $\mathbb{R}$ by $h(x + 2) = h(x)$.

This is a continuous, nowhere differentiable function.

Also note that $0 \leq |g(x)| \leq 2$ hence is bounded.

Consider

$$f_n(x) = \frac{1}{n} g(x).$$

$f_n(x)$ converges to $f(x) = 0$ uniformly, since taken $\epsilon > 0$, there exists $N > \frac{2}{\epsilon}$, $N \in \mathbb{N}$ such that for any $n \geq N$, and for any $x \in \mathbb{R}$

$$|f_n(x) - 0| \leq \frac{1}{n}|g(x)| \leq \frac{2}{n} \leq \frac{2}{N} < \epsilon.$$

f_n is nowhere differentiable for every $n \in \mathbb{N}$,

but $f(x) = 0$ and hence is differentiable everywhere.

b)

$$f_n(x) = x + n$$

is a sequence of differentiable functions.

$$f_n'(x) = 1$$

which converges uniformly to $f(x) = 1$.

$$f_n(x) = x + n$$

does not converge for any $x \in \mathbb{R}$.

c) This request is not possible.

Let $f = \lim f_n$ be not differentiable at a point x_0.

Consider the closed interval, $[x_0 - 1, x_0 + 1]$.

f_n uniformly converges to f and f_n' uniformly converges to say g on $[x_0 - 1, x_0 + 1]$.

Then we know f is differentiable and $f' = g$, contradicting the fact that f is not differentiable at x_0.

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Exercise 4 (6.3.7)

Theorem 6.3.2

Let (f_n) be a sequence of differentiable functions defined on the closed interval $[a, b]$, and assume (f_n') converges uniformly on $[a, b]$. If there exists a point $x_0 \in [a, b]$ where $f_n(x_0)$ is convergent, then (f_n) converges uniformly on $[a, b]$.

Use the Mean Value Theorem to supply a proof for Theorem 6.3.2. To get started, observe that the triangle inequality implies that, for any $x \in [a, b]$ and $m, n \in \mathbb{N}$,

$$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|.$$

Proof.
To prove that (f_n) converges uniformly on $[a, b]$, we will prove the equivalent statement, that (f_n) satisfies the Cauchy Criterion.

Given $\epsilon > 0$, we will exibit an $N \in \mathbb{N}$ such that

$$|f_m(x) - f_n(x)| < \epsilon$$

for all $n, m \geq N$ and for all $x \in [a, b]$.

Since $f_n's$ are differentiable functions hence are continuous. For any two $n, m \in \mathbb{N}$, $(f_n - f_m)$ is also a differentiable function on $[a, b]$, hence we have by Mean value Theorem,

$$|(f_n - f_m)(x) - (f_n - f_m)(x_0)| \leq |(f_n - f_m)'(\theta)||x - x_0| \leq |f_n'(\theta) - f_m'(\theta)||b - a|$$

Since (f_n') uniformly converges hence by Cauchy Criterion, there exists an $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$

$$|f_n'(\theta) - f_m'(\theta)| < \frac{\epsilon}{4(b-a)}.$$

Since $(f_n(x_0))$ converges, hence the sequence is Cauchy hence there exists $N_2 \in \mathbb{N}$, such that for all $n, m \geq N_2$ such that

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}.$$

Take $N = \max N_1, N_2$.
For all $n, m \geq N$ and for all $x \in [a, b]$, we have

$$|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|$$

$$\leq |f_n'(\theta) - f_m'(\theta)||b - a| + |f_n(x_0) - f_m(x_0)|.$$

$$\leq \frac{\epsilon}{4(b-a)}(b - a) + \frac{\epsilon}{2} < \epsilon.$$

Hence the sequence (f_n) converges uniformly.

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