**Banach Problems** Write

$$D^0 = \{f' | f : (a, b) \to \mathbb{R} \text{ is differentiable}\}$$
$$Bdd(a, b) = \{f | f : (a, b) \to \mathbb{R} \text{ is bounded}\}$$
$$C^1[a, b] = \{f|_{[a,b]} | f : (c, d) \to \mathbb{R} \text{ has a continuous derivative}\}$$
$$||f||_{sup} = \sup_x(|f(x)|) \text{ and } ||f||_{C^1} = ||f||_{sup} + ||f'||_{sup}$$

(1) Show that $(D^0(a, b) \cap Bdd(a, b), || \cdot ||_{sup})$ is a Banach space.

We first show that $D^0(a, b) \cap Bdd(a, b)$ is a $\mathbb{R}$-vector space. It is easy to check that $Bdd(a, b)$ is a $\mathbb{R}$-vector space, and $D^0(a, b)$ is a $\mathbb{R}$-vector space following from the linearity of derivatives. Therefore, their intersection is again a $\mathbb{R}$-vector space.

Next, we show $(D^0(a, b) \cap Bdd(a, b), || \cdot ||_{sup})$ is a normed vector space w.r.t. the sup-norm. We prove the three properties in Definition 13.20 in Hunter's notes. We know the sup-norm of any function is always greater than 0 by its definition, and it is finite for functions in $D^0(a, b) \cap Bdd(a, b)$ because the absolute value of a bounded function is bounded by some real number. Also, we know that for any function $f$, $||f||_{sup} = 0$ if and only if $|f(x)| = 0$ for all $x \in \mathbb{R}$ if and only if $f = 0$, so this property holds for $f \in D^0(a, b) \cap Bdd(a, b)$ as well. We just proved the first property. To show the second property, observe that for any function $f$ that is defined at $x \in \mathbb{R}$ and any scalar $k \in \mathbb{R}$, we have $|kf(x)| = |k||f(x)|$. Taking the supremum of both sides of the equality over all $x \in (a, b)$ (which we can do because $f$ is a bounded derivative function on $(a, b)$, so $f(x)$ exists for $x \in (a, b)$) gives us the second property in Definition 13.20. Now, we show the last property. Consider any $f, g \in D^0(a, b) \cap Bdd(a, b)$, we have

$$||f + g||_{sup} = \sup_x(|f(x) + g(x)|)$$

where $|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \sup_y |f(y)| + \sup_y |g(y)|$. It follows that

$$\sup_x(|f(x) + g(x)|) \leq \sup_x(\sup_y |f(y)| + \sup_y |g(y)|) = \sup_y |f(y)| + \sup_y |g(y)| = \sup_x |f(y)| + \sup_x |g(y)|$$

which proves the triangular inequality.

Last, we show the normed space $(D^0(a, b) \cap Bdd(a, b), || \cdot ||_{sup})$ is complete with respect to the sup-norm. That is, every Cauchy sequence in $D^0(a, b) \cap Bdd(a, b)$ is convergent w.r.t the sup-norm. Consider any sequence $\{f_n\}_{n=1}^{\infty} \subseteq D^0(a, b) \cap Bdd(a, b)$ that is $|| \cdot ||_{sup}$-Cauchy. With the definition we know for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $m, n > N$, we have $||f_m(x) - f_n(x)||_{sup} = \sup_{x\in(a,b)} |f_m(x) - f_n(x)| < \epsilon$. This means that if we fix any $x \in (a, b)$, then the sequence $\{f_n(x)\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$, which is also a convergent sequence by the Cauchy Criterion (Theorem 2.6.4 in Abbott). Therefore, we can define the limit function $f$ on $(a, b)$ as the pointwise limit of $f_n$. Then with a similar argument as in the proof of the Cauchy Criterion, we can show for all $n > N_\epsilon$ that $|f_n(x) - f(x)| < \epsilon$ for all $x \in (a, b)$, which implies that $f_n$ converges to $f$ under the sup-norm.

We get immediately from this that $f \in Bdd(a, b)$, because $||f||_{sup} = ||(f - f_n) + f_n||_{sup} \leq ||f - f_n||_{sup} + ||f_n||_{sup} < \infty$ for any $n > N_\epsilon$. It is left to show that $f$ is a derivative function on $(a, b)$. Our plan is to construct the primitive of $f$ on $(a, b)$ explicitly. Consider the (non-unique) sequence of primitives of $\{f_n\}_{n=1}^{\infty}$ on $(a, b)$ and denote it by $\{F_n\}_{n=1}^{\infty}$. Consider the sequence $\{F_n(x) - F_n(c)\}_{n=1}^{\infty}$ for some $c \in (a, b)$ and denote it again (with the abuse of notation) by $\{F_n(x)\}_{n=1}^{\infty}$. We see here that $F_n(c) = 0$ for all $n \in \mathbb{N}$. If we can show that $\{F_n\}$ is Cauchy w.r.t. the sup-norm on $(a, b)$, then with a similar argument as in the proof of the Cauchy Criterion, we know $F_n \to F$ for some function $F$ under the sup-norm. Then the proof complete if we can show $F$ is differentiable with its derivative coincides with $f$ on $(a, b)$. Now, we prove the sequence of functions $\{F_n\}$ is Cauchy under the sup-norm on $(a,b)$. Observe that for any $x \in (a,b)$,

$$\begin{aligned} F_m(x) - F_n(x) &= F_m(x) - F_m(c) + F_m(c) - F_n(c) + F_n(c) - F_n(x) \\ &= [F_m(x) - F_n(x)] - [F_m(c) - F_n(c)] + \underbrace{[F_m(c) - F_n(c)]}_{=0} \\ &= (x-c)\frac{[F_m(x) - F_n(x)] - [F_m(c) - F_n(c)]}{x-c} \\ &= (x-c)(f_n(\xi) - f_m(\xi)) \end{aligned}$$

for some $\xi \in (a,b)$, where the last equality follows from the MVT. It follows that

$$||F_m(x) - F_m(x)||_{sup} = ||(x-c)(f_n(\xi) - f_m(\xi))||_{sup} = \underbrace{|x-c|}_{\leq b-a}||(f_n(\xi) - f_m(\xi))||_{sup}$$

so $\{F_m\}$ is Cauchy under the sup-norm on $(a,b)$ following from the fact that $\{f_n\}$ is Cauchy under the sup-norm on $(a,b)$. So the uniform limit of $F_n$ on $(a,b)$ exists. Denote it by $F$ and observe that for any $x, y \in (a,b)$

$$\left|\frac{F(x)-F(y)}{x-y} - f(y)\right| \leq \left| \underbrace{\frac{F(x)-F(y)}{x-y} - \frac{F_n(x)-F_n(y)}{x-y}}_{\substack{= \frac{[F(x)-F_n(x)]-[F(y)-F_n(y)]}{x-y} \\ = \underbrace{f(\xi')-f_n(\xi')}_{(1)},\ \text{for some } \xi' \in (x,y)}} \right| + \underbrace{\left|\frac{F_n(x)-F_n(y)}{x-y} - f_n(y)\right|}_{(3)} + \underbrace{|f_n(y) - f(y)|}_{(2)}$$

Fix $\epsilon > 0$. Because $\{f_n\}$ converge uniformly on $(a,b)$, we can pick $N_\epsilon \in \mathbb{N}$ large enough so that the term (1) and (2) are less than $\epsilon$ for $n > N_\epsilon$. Because $F_n$ is differentiable for all $n \in \mathbb{N}$ on $(a,b)$, we can pick $\delta_\epsilon > 0$ small enough so that (3) is less than $\epsilon$ for all $x, y \in (a,b)$ with $0 < |x-y| < \delta_\epsilon$. This proves the statement. □

**Remark.** Theorem 9.13 (in Hunter's notes) alone is not suffice for the proof of the completeness of the normed space. Recall that Thm 9.13 states that any $\{f_n : A \to \mathbb{R}\}$ is $||\cdot||_{sup}$-convergent *on* $A$ if and only if it is $||\cdot||_{sup}$-Cauchy *on* $A$. Note that here the set $A$ is the *domain* of the functions. However, in the definition of Banach space, we need to show any sequence of functions $\{f_n\} \subseteq A$ that is $||\cdot||_{sup}$-Cauchy *in* $A$ is also $||\cdot||_{sup}$-convergent *in* $A$, where the set $A$ here is the *function space* that contains all $f_n$'s.

**Remark.** Theorem 9.18 (in Hunter's notes) is not suffice for the proof of the completeness of the normed space. Recall that Theorem 9.18 assumes that the sequence of primitive functions converge pointwise to the limit function, whereas in our setting we only know the behaviors of the derivative functions. And since primitives/ antiderivatives are not unique (i.e. differ up to a constanct term), we cannot assume the primitives functions converge pointwise without additional justifications.

(2) Show that $(C^1[a,b], ||\cdot||_{C^1})$ is a Banach space.

Consider any $f \in C^1[a,b]$, we know $f : [a,b] \to$ is continuous on a compact set, so $f$ is bounded. With the linearility of derivatives, we know $C^1[a,b]$ is a $\mathbb{R}$-vector space. To show $||\cdot||_{C^1}$ is a norm on $C^1[a,b]$. With a similar argument as in part (1), we can show that $(C^1[a,b], ||\cdot||_{C^1})$ is a normed vector space. It left to show that any Cauchy sequence $\{f_n\}_{n=1}^\infty \subseteq C^1[a,b]$ is convergent under the $C^1$-norm on $[a,b]$. Assume $\{f_n\} \subseteq C^1[a,b]$ is Cauchy under the $C^1$-norm. That is, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, we have $||f_m - f_n||_{C^1} = ||f_m - f_n||_{sup} + ||f'_m - f'_n||_{sup} < \epsilon$. This means that $\{f_n\}$ and $\{f'_n\}$ are Cauchy under the sup-norm on $[a,b]$. With a similar argument as in the Cauchy Criterion, we can show that both $\{f_n\}$ and $\{f'_n\}$ is convergent under the sup-norm on $[a,b]$ to their limit $f$ and $g$, respectively. Apply the Differentiable Limit Theorem, we get that $f$ is differentiable with $f' = g$ on $[a,b]$. Also, since each $f'_n$ is continuous (because $f_n \in C^1[a,b]$) and $f'_n \to f$ uniformly on $[a,b]$, with the Continuous Limit Theorem we know that $f$ is continuous on $[a,b]$ as well. This completes the proof. □