MAT 127B
HW 14 Solutions(6.4.2/6.4.3/6.4.4)

Exercise 1 (6.4.2)

Decide whether each proposition is true or false, providing a short justification or counterexample as appropriate.

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.

(b) If $0 \leq f_n(x) \leq g_n(x)$ and $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Proof.

a) True

Say $\sum_{n=1}^{\infty} g_n$ converges uniformly.

This implies the sequence

$$s_n = \sum_{k=1}^{k} g_k$$

is uniformly Cauchy which implies given $\epsilon > 0$, there exists N' such that for all $n, m > N'$, we have (assume $n > m$)

$$|s_n - s_m| < \epsilon \implies \left| \sum_{k=m+1}^{n} g_k \right| < \epsilon.$$

Hence given $\epsilon > 0$, there exists $N = N' + 1$ such that for all $n > N$, $(n > n - 1 \geq N > N')$

$$|s_n - s_{n-1}| = |g_n| < \epsilon.$$

Hence $g_n \longrightarrow 0$ uniformly.

b) True

Given $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly implies that the sequence

$$s_n = \sum_{k=1}^{n} g_k(x)$$

is uniformly Cauchy. This implies that

Given $\epsilon > 0$, there exists N' such that for all $n, m > N'$ and for all x

$$|s_n - s_m| = \left| \sum_{k=n+1}^{m} g_k(x) \right| < \epsilon.$$

Given $\epsilon > 0$ there exists $N = N'$, such that for $n, m > N$ and for all x.

$$\left| \sum_{k=1}^{m} f_k(x) - \sum_{k=1}^{n} f_k(x) \right| = \left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \left| \sum_{k=n+1}^{m} g_k(x) \right| < \epsilon.$$

Note that the second to last inequality is true since $0 \leq f_k(x)$ and $0 \leq g_k(x)$

c) False

Let $A = [-1, 1]$ and

$$f_1(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$f_2(x) = \begin{cases} \frac{-1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$f_n(x) = 0 \ \text{ (for } n \geq 3).$$

Given $\epsilon > 0$, take $N = 3$.

For all $n \geq N$ and for all $x \in A$

$$\left| \sum_{k=1}^{n} f_k(x) - 0 \right| = 0 < \epsilon.$$

Hence f_n converges to 0 uniformly but f_1 is not bounded. Hence it is impossible to get M_1.

Hence the above proposition is not possible.

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Exercise 2 (6.4.3)

a) Show that

$$g(x) = \sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$$

is continuous on all of $\mathbb{R}$.

b) The function g was cited in Section 5.4 as an example of a continuous nowhere differentiable function. What happens if we try to use Theorem 6.4.3 to explore whether g is differentiable?

Proof.

a) Consider the sequence

$$s_n(x) = \sum_{k=0}^{n} \frac{\cos(2^k x)}{2^k}$$

Need to prove that $s_n \longrightarrow g$ uniformly (or the sequence $\{s_n\}$) is uniformly Cauchy) and s_n is continuous for every n.

Given $\epsilon > 0$ take N such that $2^N > \epsilon$.

Given $n, m > N$, $(n < m)$ we have

$$|s_n - s_m| = \left| \sum_{k=n+1}^{m} \frac{\cos(2^k x)}{2^k} \right| \leq \sum_{k=n+1}^{m} \left| \frac{\cos(2^k x)}{2^k} \right|$$
$$\leq \sum_{k=n+1}^{m} \left| \frac{1}{2^k} \right| < \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n} < \frac{1}{2^N} < \epsilon.$$

Since this is true for all x hence s_n is uniformly Cauchy, hence converges uniformly.

Given k,

$$f_k(x) = \frac{\cos(2^k x)}{2^k}$$

is a continuous function on x. Finite sum of continuous functions are continuous, hence each s_n is a continuous function.

b) If we try to apply Theorem 6.4.3, then we have

$$f_k(x) = \frac{\cos(2^k x)}{2^k}$$

and each $f_k(x)$ are differentiable functions with

$$f_k'(x) = -\sin(2^k x).$$

Moreover if $x = 0$, then

$$\sum_{k=0}^{\infty} f_k(0) = \sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

We need to show that for some $x_0 \in \mathbb{R}$,

the series does not converge, which implies that Theorem 6.4.3 would fail.

Consider $x_0 = \frac{2\pi}{3}$.

$$\sin(2^k x_0) = \sin(2\pi/3) = \frac{\sqrt{3}}{2} \text{ if k is even;} \qquad \sin(2^k x_0) = \sin(4\pi/3) = -\frac{\sqrt{3}}{2} \text{ if k is odd.}$$

$$s_n = \sum_{k=0}^{n} -\sin(2^k x_0)$$

If n is odd, $s_n = 0$.

If n is even $s_n = \sqrt{3}/2$.

As we can see, 0 is a limit point of the series, but the series does not converge to 0.

Hence we cannot apply Theorem 6,4,3

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Exercise 3 (6.4.4)

Define

$$g(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(1 + x^{2n})}.$$

Find the values of x where the series converges and show that we get a continuous function on this set.

Proof.
Let us consider the pointwise convergence of the functions

$$f_n(x) = \frac{x^{2n}}{1 + x^{2n}}.$$

Let $h(x)$ be the pointwise convergence of $f_n(x)$.
Then

$$h(x) = \begin{cases} 0, & \text{if } |x| < 1 \\ \frac{1}{2}, & \text{if } x = \pm 1 \\ 1, & \text{if } |x| > 1. \end{cases}$$

We know that if $\sum_{n=0}^{\infty} f_n(x)$ converges then for each x, $f_n(x) \longrightarrow 0$.
Hence the necessary condition shows $|x| < 1$.
Now given any x, such that $|x| < 1$, also noting that $1 + x^{2n} > 1$

$$\left| \sum_{n=0}^{\infty} \frac{x^{2n}}{1 + x^{2n}} \right| \leq \left| \sum_{n=0}^{\infty} x^{2n} \right| = \frac{1}{1 - x^2}.$$

Since the sequence

$$s_n = \sum_{k=0}^{n} f_k(x)$$

is an increasing sequence and bounded above hence the sequence $\{s_n(x)\}$ converges.

Now we will show that the function $g(x)$ is also continuous on $(-1, 1)$.
Consider the functions $f_n(x)$. Each of these functions are continuous on $(-1, 1)$, hence $s_n = \sum_{k=0}^{n} f_n(x)$ a finite sum of continuous functions hence is continuous on $(-1, 1)$.

We need to prove that $g(x)$ is continuous at every point $x \in (-1, 1)$.
Let x_0 be any point in $(-1, 1)$.
Let

$$y_1 = \frac{x + 1}{2}; \qquad y_2 = \frac{x - 1}{2}.$$

Then we have $-1 < y_2 < x < y_1 < 1$.
Hence consider the set $A_x = [y_2, y_1]$.

We will show that $\{s_n\}$ uniformly converges in A_x. (The sequence s_n is uniformly Cauchy in A_x)
Given $\epsilon > 0$ take N such that

$$\frac{\max\{|y_1|, |y_2|\}^{2N}}{1 - (\max\{|y_1|, |y_2|\})^2} < \epsilon.$$

Let $\max\{|y_1|, |y_2|\} = k$.
Since $k < 1$, hence $k^{2n} \longrightarrow 0 \implies k^{2n}/1 - k^2 \longrightarrow 0$. This implies there exists an N, which satisfies the above inequality.

Then for any $n, m > N$, $(n < m)$ we have

$$|s_n - s_m| = \sum_{k=n+1}^{m} \frac{x^{2k}}{1 + x^{2k}} < \sum_{k=n+1}^{m} x^{2k} \leq \sum_{k=N}^{\infty} x^{2n} = \frac{x^{2N}}{1 - x^2} < \epsilon.$$

Hence $\{s_n\}$ is uniformly Cauchy in A_x, and each function $\{s_n\}$ is continuous implies $g(x)$ is continuous at x.
Since $x \in (-1, 1)$ was arbitrary to begin with, hence $g(x)$ is continuous for every $x \in (-1, 1)$.
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