6.5.2 Find suitable coefficients (a_n) so that the resulting power series $\sum a_n x^n$ has the given properties, or explain why such a request is impossible.

(a) Converges for every value of $x \in \mathbb{R}$

Consider $a_n = \frac{1}{n^n}$. Wlog, consider $x \in \mathbb{R}_+$, we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left(\frac{x}{n}\right)^n = \underbrace{\sum_{n \le x+1} \left(\frac{x}{n}\right)^n}_{<\infty} + \underbrace{\sum_{n > x+1} \left(\frac{x}{n}\right)^n}_{\le \sum_{n=0}^{\infty} (\frac{x}{\lceil x+1 \rceil})^n < \infty} < \infty$$

where $\lceil x+1 \rceil$ is the smallest integer that is greater than $x+1$. And the convergence of the second term follows from the geometric series. □

(b) Diverges for every value of $x \in \mathbb{R}$?

This is impossible. Because $\sum a_n x^n$ converges at $x = 0$ for any (a_n).

(c) Converges absolutely for all $x \in [-1, 1]$ and diverges off of this set.

Consider $a_n = \frac{1}{n^2}$. We then have when $x = 1$ that

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n^2} < \infty$$

However, when evaluated at $x = 1 + \epsilon$ for any $\epsilon > 0$, we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{(1+\epsilon)^n}{n^2}$$

If we can show that $(1+\epsilon)^n > n$ for n sufficiently large, then we know the tail of this series is bounded below by the tail of $\sum_{n=0}^{\infty} \frac{1}{n} = \sum_{n=0}^{N} \frac{1}{n}$, which goes to ∞ as $N \to \infty$. Consider the binomial expansion of $(1+\epsilon)^n$, we get

$$(1+\epsilon)^n = 1 + n\epsilon + \frac{n(n-1)}{2}\epsilon^2 + \sum_{i=3}^{n} \binom{n}{i} \epsilon^i$$

We pick $n \in \mathbb{N}$ large enough so that $\frac{(n-1)\epsilon^2}{2} > 1$, then $(1+\epsilon)^n \ge \frac{n(n-1)\epsilon^2}{2} > n$. This finishes the proof. □

6.5.4 (Term-by-term Antidifferentiation). Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$.

(a) Show

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

is defined on $(-R, R)$ and satisfies $F'(x) = f(x)$.

For any $x \in (-R, R)$, we can find $|x| < x_0 < R$ such that $\sum a_n x^n$ converges at x_0. By Theorem 6.5.1, we have $\sum a_n x^n$ converges absolutely at x. It follows that

$$\sum_{n=0}^{\infty} \left|\frac{a_n}{n+1}\right| x^{n+1} \le x \sum_{n=0}^{\infty} \underbrace{\left|\frac{a_n}{n+1}\right|}_{\le |a_n|} x^n \le x \sum_{n=0}^{\infty} |a_n| x^n < \infty$$

so $F(x)$ converges absolutely (hence converges) on $(-R, R)$. The linearity of derivatives then yields

$$F'(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} \frac{d}{dx}(x^{n+1}) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(n+1)x^n = f(x)$$

□

(b) Antiderivatives are not unique. If g is an arbitrary function satisfying $g'(x) = f(x)$ on $(-R, R)$, find a power series representation for g.

We know from part (a) that $F(x)$ is a function satisfying $F'(x) = f(x)$ on $(-R, R)$. Recall from Corollary 5.3.4 that two functions with the same derivative on their domain (in $\mathbb{R}$) differs only by a constant term. This means that any function g satisfying $g'(x) = f(x)$ on $(-R, R)$ will be of the form $g(x) = F(x) + c$ where $c \in \mathbb{R}$. □

6.5.8

(a) Show that the power series representation are unique. If we have

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$

for all x in an interval $(-R, R)$, prove that $a_n = b_n$ for all $n = 0, 1, 2, \cdots$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$. Differentiating both $A(x)$ and $B(x)$ w.r.t. x, we get

$$A^{(k)}(x) = \sum_{n=k}^{\infty} k! a_n x^{n-k} = B^{(k)}(x) = \sum_{n=k}^{\infty} k! b_n x^{n-k}$$

for $k = 0, 1, 2, \cdots$. Evaluate the above equations at $x = 0$, we get for all $k = 0, 1, 2, \cdots$ that

$$k! a_k = k! b_k \Leftrightarrow a_k = b_k$$

□

(b) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converge on $(-R, R)$, and assume $f'(x) = f(x)$ for all $n \in (-R, R)$ and $f(0) = 1$. Deduce the values of a_n.

We have from the statement that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

Equating the coefficients gives us the recursive relations $a_n = (n+1)a_{n+1}$ for all $n = 0, 1, 2, \cdots$ with the initial term $a_0 = f(0) = 1$. It follows that for $N \in \mathbb{N}$ that $a_n = \frac{1}{n} a_{n-1}$. Hence, $a_n = \frac{1}{n!}$ for $n = 0, 1, 2, \cdots$. □