MAT 127B
HW 16 Solutions(5.3.8/6.2.8/6.4.8)

Exercise 1 (5.3.8)

Assume f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x\to 0} f'(x) = L$, show $f'(0)$ exists and equals L

Proof.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x}.$$

Since f is continuous, hence $\lim_{x\to 0} f(x) = f(0)$. Hence we can use $L'Hospital'$s Rule.

$$\lim_{x\to 0} \frac{(f(x) - f(0))'}{(x)'} = \lim_{x\to 0} \frac{f'(x)}{1} = L \implies f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = L$$

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Exercise 2 (6.2.8)

Let (g_n) be a sequence of continuous functions that converges uniformly to g on a compact set K. If $g(x) \neq 0$ on K, show $(1/g_n)$ converges uniformly on K to $1/g$.

Proof.
Since $g_n(x)$ are continuous and uniformly converges to $g(x)$ hence $g(x)$ is continuous on K. Since K is compact, and $g(x)$ is continuous, hence $g(x)$ is bounded implies $m < |g(x)| < M$ for some m, M.

We know that a continuous function defined on a compact set achieves its supremum and infimum.
This implies there exists $x \in K$ such that $|g(x)| = m$.
Since $g(x) \neq 0$ in K, hence $m > 0$.

Since $g_n(x)$ converge uniformly to $g(x)$, hence there exists N_1 such that for all $x \in K$ and for all $n \geq N_1$
$|g(x) - g_n(x)| < m - m_1$ implies
$m \leq |g(x)| \leq |g_n(x)| + m - m_1$
implies

$$|g_n(x)| \geq m_1$$

for all $n \geq N_1$.
Given $\epsilon > 0$, there exists N_2 such that for all $x \in K$ and $n \geq N_2$

$$|g_n(x) - g(x)| < \epsilon(m_1)(m)$$

$$\left| \frac{1}{g_n(x)} - \frac{1}{g(x)} \right| = \frac{|g(x) - g_n(x)|}{|g_n(x)||g(x)|}.$$

Choose $N = \max\{N_1, N_2\}$.
For $n \geq N \geq N_1$ we have

$$|g_n(x)| > m_1, \implies |g_n(x)||g(x)| > (m_1)(m) \implies \frac{1}{|g_n(x)||g(x)|} < \frac{1}{(m_1)(m)}$$

Hence for all $x \in K$ and for all $n \geq N$, we have

$$\left| \frac{1}{g_n(x)} - \frac{1}{g(x)} \right| = \frac{|g(x) - g_n(x)|}{|g_n(x)||g(x)|} < \frac{|g(x) - g_n(x)|}{(m_1)(m)} < \epsilon.$$

Hence $(1/g_n)$ converge uniformly on K to $1/g$. ∎

Exercise 3 (6.4.8)

Consider the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(x/k)}{k}$$

.
Where is f defined? Continuous? Differentiable? Twice-differentiable?

Proof.
$f(x)$ is defined on $\mathbb{R}$.

Fix any $x \in \mathbb{R}$.
For any k, we have the Taylor's theorem which states,

$$\sin\left(\frac{x}{k}\right) = \frac{x}{k} - \frac{\sin(\xi_k)}{2}\frac{x^2}{k^2}.$$

Hence

$$\left|\sum_{k=1}^{\infty}\frac{\sin(x/k)}{k}\right| \le \sum_{k=1}^{\infty}\left|\frac{\sin(x/k)}{k}\right| = \sum_{k=1}^{\infty}\left|\frac{\frac{x}{k}-\frac{\sin(\xi_k)}{2}\frac{x^2}{k^2}}{k}\right| = \sum_{k=1}^{\infty}\left|\frac{x}{k^2}-\frac{\sin(\xi_k)}{2}\frac{x^2}{k^3}\right|$$

$$\sum_{k=1}^{\infty}\left|\frac{x}{k^2}-\frac{\sin(\xi_k)}{2}\frac{x^2}{k^3}\right| \le \sum_{k=1}^{\infty}\left|\frac{x}{k^2}\right| + \sum_{k=1}^{\infty}\frac{|\sin(\xi_k)|}{2}\left|\frac{x^2}{k^3}\right| \le \sum_{k=1}^{\infty}\left|\frac{x}{k^2}\right| + \sum_{k=1}^{\infty}\frac{1}{2}\left|\frac{x^2}{k^3}\right|$$

Since both the series converge hence is bounded say by M.
This implies

$$\left|\sum_{k=1}^{\infty}\frac{\sin(x/k)}{k}\right| \le M.$$

Choose $N > 2x/\pi$ ($N > -2x/\pi$) such that for all $n \ge N$, $(x/n) < \pi/2$, $(x/n > -\pi/2)$ for $x > 0$ ($x < 0$ respectively)
hence if $s_n = \sum_{k=1}^{n}\frac{\sin(x/k)}{k}$, then the sequence $\{s_n\}$ is increasing (decreasing) for $n > N$ and is bounded hence converges.

If $x = 0$, $f(0) = 0$.
Hence $f(x)$ is defined on all of $\mathbb{R}$.

Given $x_0 \in \mathbb{R}$, consider $A_{x_0} = [-x_0 - 1, x_0 + 1]$.
Let $|x| < M_{x_0}$ for $x \in A_{x_0}$.
Since

$$\sum_{k=1}^{\infty}\frac{1}{k^2}; \qquad \sum_{k=1}^{\infty}\frac{1}{k^3}$$

converge hence the tail sum converges to 0 or in other words,
Given $\epsilon > 0$, there exists N_1 and N_2, such that

$$\text{for } n \ge N_1,\ \sum_{k=n}^{\infty}\frac{1}{k^2} < \frac{\epsilon}{2M_{x_0}}; \qquad \text{and for } n \ge N_2,\ \sum_{k=n}^{\infty}\frac{1}{k^3} < \frac{\epsilon}{2M_{x_0}^2}.$$

Let $N = \max\{N_1, N_2\}$.
Then for any $n \ge N$, and for all $x \in A_{x_0}$, we have

$$\left|f(x) - \sum_{k=1}^{n}\frac{\sin(x/k)}{k}\right| = \left|\sum_{k=n+1}^{\infty}\frac{\sin(x/k)}{k}\right|$$

$$\leq \sum_{k=n+1}^{\infty} \left| \frac{\sin(x/k)}{k} \right| = \sum_{k=n+1}^{\infty} \left| \frac{\frac{x}{k} - \frac{\sin(\xi_k)}{2}\frac{x^2}{k^2}}{k} \right| = \sum_{k=n+1}^{\infty} \left| \frac{x}{k^2} - \frac{\sin(\xi_k)}{2}\frac{x^2}{k^3} \right|$$

$$\sum_{k=n+1}^{\infty} \left| \frac{x}{k^2} - \frac{\sin(\xi_k)}{2}\frac{x^2}{k^3} \right| \leq \sum_{k=n+1}^{\infty} \left| \frac{x}{k^2} \right| + \sum_{k=n+1}^{\infty} \frac{|\sin(\xi_k)|}{2} \left| \frac{x^2}{k^3} \right|$$

$$\leq \sum_{k=n+1}^{\infty} \left| \frac{x}{k^2} \right| + \sum_{k=n+1}^{\infty} \frac{1}{2} \left| \frac{x^2}{k^3} \right| = |x| \sum_{k=n+1}^{\infty} \frac{1}{k^2} + |x|^2 \sum_{k=n+1}^{\infty} \frac{1}{2k^3} < |x| \frac{\epsilon}{2M_{x_0}} + |x|^2 \frac{\epsilon}{2M_{x_0}^2}$$

Since $x \in A_{x_0}$ and $|x| < M_{x_0}$ hence,

$$|x| \frac{\epsilon}{2M_{x_0}} + |x|^2 \frac{\epsilon}{2M_{x_0}^2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Note since N is independent of $x \in A_{x_0}$, hence the series converges uniformly.
Moreover if

$$s_n = \sum_{k=1}^{n} \frac{\sin(x/k)}{k},$$

then s_n is continuous since it is a finite sum of continuous functions in x.

$s_n(x) \longrightarrow f(x)$ uniformly in A_{x_0} and each s_n is continuous, hence $f(x)$ is continuous in A_{x_0}.
This implies f is continuous at the point x_0.
Since x_0 was arbitrarily chosen hence the function $f(x)$ is continuous in $\mathbb{R}$.

Each $f_k(x) = \frac{\sin(x/k)}{k}$ is differentiable with

$$f_k'(x) = \frac{\cos(x/k)}{k^2}$$

Consider

$$g(x) = \sum_{k=1}^{\infty} \frac{\cos(x/k)}{k^2}$$

Choose $N_1 > \frac{2M_{x_0}}{\pi}$.
Then for any $n \geq N_1$,

$$t_n = \sum_{k=1}^{n} \frac{\cos(x/k)}{k^2}$$

t_n is increasing.

$$|t_n| \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

hence bounded.
This implies $g(x)$ exists for all x.
Moreover the convergence is uniform in A_{x_0} since

$$|g(x) - t_n| = \left| \sum_{k=n+1}^{\infty} \frac{\cos(x/k)}{k^2} \right| \leq \sum_{k=n+1}^{\infty} \left| \frac{\cos(x/k)}{k^2} \right| \leq \sum_{k=n+1}^{\infty} \frac{1}{k^2}$$

Given $\epsilon > 0$ and since $\sum_{k=1}^{\infty} 1/k^2$ converges, hence tail sums converge to 0, so there exists an $N > N_1$(independent of $x \in A_{x_0}$) such that for all $n \geq N$

$$\sum_{k=n}^{\infty} \frac{1}{x^2} < \epsilon$$

Hence the convergence is uniform in A_{x_0}.
Moreover $\sum_{k=1}^{\infty} f_k(x)$ converge for $x = x_0$.
Hence $f(x)$ is differentiable with the derivative $f'(x) = g(x)$. in A_{x_0}.
Hence f is differentiable at x_0.
Since x_0 is arbitrary hence f is differentiable for all $x \in \mathbb{R}$.

Let

$$h(x) = \sum_{k=1}^{\infty} \left(\frac{\cos(x/k)}{k^2} \right)' = \sum_{k=1}^{\infty} \frac{\sin(x/k)}{k^3}$$

Replacing $f(x)$ with $g(x)$ and $g(x)$ with $h(x)$, we can do the same trick to conclude that the function $f(x)$ is twice differentiable.

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