7.2.2 Consider $f(x) = 1/x$ over the interval $[1, 4]$. Let P be the partition consisting of the points $\{1, 3/2, 2, 4\}$.

(a) Compute $L(f, P)$, $U(f, P)$, and $U(f, P) - L(f, P)$.

With the same notations as in the textbook, we have

$$L(f,P) = \sum_{k=1}^{3} m_k \Delta x_k = f\Big(\frac{3}{2}\Big)\Big(\frac{3}{2} - 1\Big) + f(2)\Big(2 - \frac{3}{2}\Big) + f(4)(4-2) = \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{4}\cdot 2 = \frac{13}{12}$$

$$U(f,P) = \sum_{k=1}^{3} M_k \Delta x_k = f(1)\Big(\frac{3}{2} - 1\Big) + f\Big(\frac{3}{2}\Big)\Big(2 - \frac{3}{2}\Big) + f(2)(4-2) = 1\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{2}\cdot 2 = \frac{11}{6}$$

$$U(f,P) - L(f,P) = \frac{11}{6} - \frac{13}{12} = \frac{3}{4}$$

(b) What happens to the value of $U(f, P) - L(f, P)$ when we add the point 3 to the partition?

$$U(f,P) - L(f,P) = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{2}{3} + 1\cdot\frac{1}{2} + 1\cdot\frac{1}{3} - \frac{1}{2}\cdot\frac{2}{3} - \frac{1}{2}\cdot\frac{1}{2} - 1\cdot\frac{1}{3} - 1\cdot\frac{1}{4} = \frac{1}{2}$$

(c) Find a partition P' of $[1, 4]$ for which $U(f, P') - L(f, P') < \frac{2}{5}$.

Consider the partition $P = \{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\}$. Then we have

$$\begin{aligned} U(f,P) - L(f,P) &= \frac{1}{2}\Big[\Big(1 - \frac{3}{2}\Big) + \Big(\frac{3}{2} - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{5}{2}\Big) + \Big(\frac{5}{2} - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{7}{2}\Big) + \Big(\frac{7}{2} - \frac{1}{4}\Big)\Big] \\ &= \frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8} < \frac{2}{5} \end{aligned}$$

7.2.4 Let g be bounded on $[a, b]$ and assume there exists a partition P with $L(g, P) = U(g, P)$. Describe g. Is g necessarily continuous? Is it integrable? If so, what is the value of $\int_a^b g$?

We show that g is a constant on $[a, b]$. Rewriting the statement, we can find for $g : [a, b] \to \mathbb{R}$ a partition $P = \{a = x_0 < \cdots < x_n = b\}$ of $[a, b]$ such that for all $\epsilon > 0$,

$$U(g,P) - L(g,P) = \sum_{i=1}^{n} (M_k - m_k)\Delta x_k < \epsilon$$

Assume fsc that g is not a constant on $[x_{k-1}, x_k]$, then pick $0 < \epsilon < (M_k - m_k)\Delta x_k$ and we get a contradiction. This means that g is a constant on each $[x_{k-1}, x_k]$ for $k = 1, \cdots, n$ and must be a constant on $[a, b]$ since each two adjacent intervals share an end point. Therefore, g is continuous and integrable with $\int_a^b g = g(a)[b - a]$.

7.2.5 Assume that, for each n, f_n is an integrable function on $[a, b]$. If $(f_n) \to f$ uniformly on $[a, b]$, prove that f is also integrable on this set. (We will see that that this conclusion does not necessarily follow if the convergence is pointwise.)

Fix $\epsilon > 0$. Because $f_n \to f$ uniformly on $[a, b]$, we can find $N_\epsilon \in \mathbb{N}$ such that $|f_{N_\epsilon}(x) - f(x)| < \epsilon$ for all $x \in [a, b]$. Because f_{N_ϵ} is integrable on $[a, b]$, there is a partition P of $[a, b]$ such that $U(f_{N_\epsilon}, P) - L(f_{N_\epsilon}, P) < \epsilon$. A little computation then shows that $U(f, P) \le U(f_{N_\epsilon}, P) + (b - a)\epsilon$ and $L(f, P) \ge L(f_{N_\epsilon}, P) - (b - a)\epsilon$. It follows that $U(f, P) - L(f, P) \le [2(b - a) + 1]\epsilon$, so f is integrable on $[a, b]$. $\square$