MAT 127B
HW 18 Solutions(7.2.3/7.2.7/7.3.1)

Exercise 1 (7.2.3)

(a) Prove that a bounded function f is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying:

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$$

and in this case
$\int_a^b f(x) = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$.

(b) For each n, let P_n be the partition of $[0, 1]$ into n equal subinterval. Find formulas for $U(f, P_n)$ and $L(f, P_n)$ if $f(x) = x$. The formula $1 + 2 + 3 + \cdots + n = n(n + 1)/2$ will be useful.

(c) Use the sequential criterion for integrability from (a) to show directly that $f(x) = x$ is integrable on $[0, 1]$ and compute $\int_0^1 f(x)$.

Proof.

a) $\Rightarrow$

Say f be a bounded function which is integrable.

Since f is bounded, hence $U(f, P), L(f, P)$ exists for every partition and since f is integrable we have $U(f) = L(f)$,

$$U(f) = \inf_P U(f, P); L(f) = \sup_P L(f, P).$$

Claim: If P_1, P_2 be two partitions, then $P = P_1 \cup P_2$ is a finer partition with

$$U(f, P) \leq U(f, P_i) \text{ and } L(f, P) \geq L(f, P_i) \text{ for } i = 1, 2.$$

We will prove for $i = 1$. (It follows similarly for $i = 2$.)

Let $a_i, a_{i+1} \in P_1$.

If there exists $b_s \in P_2$ such that $a_i < b_j < b_{j+1} < \dots b_k < a_{i+1}$, then in $U(f, P)$, for the interval $[a_i, a_{i+1}]$ we have

$$M_j(b_j - a_i) + \sum_{l=j+1}^{k} M_l(b_l - b_{l-1}) + M_{k+1}(a_{i+1} - b_k),$$

where

$$M_l = \sup\{f(x)|x \in [b_{l-1}, b_l]\} \leq \sup\{f(x)|x \in [a_i, a_{i+1}]\} = M$$

and similarly $M_j = \sup\{f(x)|x \in [a_i, b_j]\} \leq M$, $M_{k+1} = \sup\{f(x)|x \in [b_k, a_{i+1}]\} \leq M$.

Hence

$$M_j(b_j - a_i) + \sum_{l=j+1}^{k} M_l(b_l - b_{l-1}) + M_{k+1}(a_{i+1} - b_k) \leq M(a_{i+1} - a_i).$$

This is true for every consecutive pair of points in P_1.

This implies

$$U(f, P) \leq U(f, P_1).$$

For the lower sums change every inequality. Then the same logical implications hold true and hence

$$L(f, P) \geq L(f, P_1).$$

Similarly true for P_2.

Given $n \in \mathbb{N}$ consider $U(f) + \frac{1}{n}$ and $L(f) - \frac{1}{n}$.

Then there exists P_n^1 and P_n^2 such that

$$U(f) \leq U(f, P_n^1) \leq U(f) + \frac{1}{n}$$

and

$$L(f) \geq L(f, P_n^2) \geq L(f) - \frac{1}{n}$$

Consider $P_n = P_n^1 \cup P_n^2$.

Then

$$U(f) \leq U(f, P_n) \leq U(f, P_n^1) \leq U(f) + \frac{1}{n}$$

and

$$L(f) \geq L(f, P_n) \geq L(f, P_n^2) \geq L(f) - \frac{1}{n}$$

Hence,

$$\lim_{n\to\infty} U(f) \leq \lim_{n\to\infty} U(f, P_n) \leq \lim_{n\to\infty} U(f) + \frac{1}{n} = U(f)$$

$\implies U(f) = \lim_{n\to\infty} U(f, P_n)$
and

$$\lim_{n\to\infty} L(f) \geq \lim_{n\to\infty} L(f, P_n) \geq \lim_{n\to\infty} L(f) - \frac{1}{n} = L(f).$$

$\implies L(f) = \lim_{n\to\infty} L(f, P_n)$
Since f is integrable,

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = U(f) - L(f) = 0.$$

$\Leftarrow$

Say there exists P_n such that

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0.$$

We know

$$U(f) \leq U(f, P_n) \text{ and } L(f) \geq L(f, P_n)$$

for every n,
hence for every n

$$0 \leq U(f) - L(f) \leq U(f, P_n) - L(f, P_n).$$

$$0 \leq U(f) - L(f) \leq \lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0.$$

This implies

$$U(f) = L(f)$$

and hence f is integrable.

Moreover we have seen,

$$\lim_{n\to\infty} U(f,P_n) = U(f) = \int_a^b f = L(f) = \lim_{n\to\infty} L(f,P_n).$$

b) Let P_n be as stated.

$P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{k}{n}, \dots, 1\}$

$f(x) = x$. For each interval $[\frac{i}{n}, \frac{i+1}{n}]$, with $i = 0, 1, \dots, n-1$

$$M_i = \sup\left\{ f(x) | x \in \left[\frac{i}{n}, \frac{i+1}{n}\right]\right\} = \frac{i+1}{n}$$

and

$$m_i = \inf\left\{ f(x) | x \in \left[\frac{i}{n}, \frac{i+1}{n}\right]\right\} = \frac{i}{n}$$

Hence

$$U(f,P_n) = \sum_{i=0}^{n-1} M_i\left(\frac{1}{n}\right) = \left(\frac{1}{n^2}\right)\sum_{i=0}^{n-1} i+1 = \frac{(n)(n+1)}{2n^2}$$

and

$$L(f,P_n) = \sum_{i=0}^{n-1} m_i\left(\frac{1}{n}\right) = \left(\frac{1}{n^2}\right)\sum_{i=0}^{n-1} i = \frac{(n)(n-1)}{2n^2}.$$

c) Consider the sequence P_n as for part b)

$$\lim_{n\to\infty}[U(f,P_n) - L(f,P_n)] = \lim_{b\to\infty}\frac{2n}{2n^2} = \lim_{n\to\infty}\frac{1}{n} = 0.$$

Since $f(x) = x$ is bounded in $[0,1]$ hence f is integrable.

Moreover,

$$\int_0^1 f(x) = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty}\frac{n^2+n}{2n^2} = \frac{1}{2}.$$

$\blacksquare$

Exercise 2 (7.2.7)

Let $f : [a, b] \longrightarrow \mathbb{R}$ be increasing on the set $[a, b]$ (i.e., $f(x) \leq f(y)$ whenever $x < y$). Show that f is integrable on $[a, b]$.

Proof.

Consider the partition $P_n = \left\{a, a + \frac{(b-a)}{n}, \ldots, a + \frac{k(b-a)}{n} \ldots, b\right\}$.

Given any interval, $(k = 0, 1, \ldots n - 1)$

$$A_k = \left[a + \frac{k(b-a)}{n}, a + \frac{(k+1)(b-a)}{n}\right]$$

$$M_k = \sup\{f(x)|x \in A_k.\} = f\left(a + \frac{(k+1)(b-a)}{n}\right)$$

and

$$m_k = \sup\{f(x)|x \in A_k.\} = f\left(a + \frac{(k)(b-a)}{n}\right).$$

Hence

$$U(f, P_n) = \sum_{k=0}^{n-1} f\left(a + \frac{(k+1)(b-a)}{n}\right) \frac{(b-a)}{n}$$

$$L(f, P_n) = \sum_{k=0}^{n-1} f\left(a + \frac{(k)(b-a)}{n}\right) \frac{(b-a)}{n}$$

$$U(f, P_n) - L(f, P_n) = \frac{(b-a)}{n}(f(b) - f(a)).$$

Hence

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0.$$

Also

$$f(a) \leq f(x) \leq f(b)$$

for $x \in [a, b]$
implying by the previous exercise that f is a bounded function, hence f is integrable.

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Exercise 3 (7.3.1)

Consider the function

$$h(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1 \\ 2, & \text{if } x = 1 \end{cases}$$

over the interval $[0, 1]$.

a) Show that $L(f, P) = 1$ for every partition P of $[0, 1]$.

b) Construct a partition P for which $U(f, P) < 1 + 1/10$.

c) Given $\epsilon > 0$, construct a partition P for which $U(f, P_\epsilon) < 1 + \epsilon$.

Proof.

a) Let $P = \{0 = x_0, x_1, x_2 \ldots, x_n = 1\}$ is a partition.
Then for $k = 1, 2, \ldots, n$

$$m_k = \inf\{f(x) | x \in [x_{k-1}, x_k]\} = 1$$

Hence

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n}(x_k - x_{k-1}) = x_n - x_0 = 1.$$

b) Let $n \in \mathbb{N}$ and consider the partition
$P_n = \left\{0, \frac{1}{n}, \ldots, \frac{k}{n} \ldots, 1\right\}$.

$$M_k = \sup\left\{f(x) | x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]\right\} = \begin{cases} 1 & \text{if } 1 \leq k \leq n-1 \\ 2 & \text{if } k = n \end{cases}$$

Hence

$$U(f, P_n) = \sum_{k=1}^{n} M_k\left(\frac{1}{n}\right) = \sum_{k=1}^{n-1} M_k\left(\frac{1}{n}\right) + \frac{2}{n} = \frac{n-1}{n} + \frac{2}{n} = \frac{n+1}{n} = 1 + \frac{1}{n}.$$

Let $n = 11$, and $P = P_n$.
Then $U(f, P) = 1 + \frac{1}{11} < 1 + \frac{1}{10}$

c) Given $\epsilon > 0$, choose N such that $\frac{1}{N} < \epsilon$,
Then from part b) choose $P_\epsilon = P_N$. Then

$$U(f, P_\epsilon) = 1 + \frac{1}{N} < 1 + \epsilon.$$

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