MAT 127B
HW 20 Solutions(7.3.5/7.3.6/7.3.7)

Exercise 1 (7.3.5)

Provide an example or give a reason why the request is impossible.

(a) A sequence $(f_n) \longrightarrow f$ pointwise, where each f_n has at most a finite number of discontinuities but f is not integrable.

(b) A sequence $(g_n) \longrightarrow g$ uniformly where each g_n has at most a finite number of discontinuities and g is not integrable.

(c) A sequence $(h_n) \longrightarrow h$ uniformly where each h_n is not integrable but h is integrable.

Proof.

a) Consider an enumeration of the rationals, say $\{r_1, r_2, \dots\} = \mathbb{Q}$.

Define

$$f_n(x) = \begin{cases} 1 & \text{if } x = r_1, r_2, , \dots, r_n \\ 0 & \text{otherwise} \end{cases}$$

Then $f_n(x) \longrightarrow f(x)$ where

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

which is the Dirichlet function and we know that $f(x)$ is not integrable. On the other hand $f_n(x)$ has finitely many discontinuities namely, $x = r_1, r_2, \dots, r_n$.

Moreover, pointwise we have the convergence since

$$f_n(x) \longrightarrow f(x).$$

For $x \in \mathbb{Q}$, $x = r_N$ for some N, and for every $n \geq N$,

$f_n(x) = 1$ hence $f(x) = 1$.

For $x \notin \mathbb{Q}$, $f_n(x) = 0$ for all n hence $f(x) = 0$.

b) Let $g_n : [1, \infty] \longrightarrow \mathbb{R}$. be a function defined by,

$$g_n(x) = \begin{cases} \frac{1}{x} & \text{if } x \leq n \\ 0 & \text{if } x > n \end{cases}$$

Now $g_n(x)$ converges to $g(x) = \frac{1}{x}$.

$$|g_n(x) - g(x)| = \begin{cases} 0 & \text{if } x \leq n \\ \frac{1}{x} & \text{if } x > n \end{cases}$$

Hence we have

$$|g_n(x) - g(x)| \leq \frac{1}{n}.$$

Hence given $\epsilon > 0$, there exists $N > \frac{1}{\epsilon}$, then for $n \geq N$, we have

$$|g_n(x) - g(x)| < \epsilon,$$

hence $g_n(x)$ converges to $g(x)$ uniformly.

Each $g_n(x)$ has 1 discontinuity point.

$$\int_1^{\infty} g(x) = \lim_{b \to \infty} \int_1^b \frac{1}{x} = \lim_{b \to \infty} ln(b)$$

diverges.

Hence $g(x)$ is not integrable.

If g_n is defined on a closed interval $[a, b]$, and $g_n \longrightarrow g$ uniformly on $[a, b]$ then g is integrable.

g_n has a finite number of discontinuities hence g_n is integrable.

We have

$$\int_a^b g_n(x) \pm \epsilon = \int_a^b \left[g_n(x) \pm \frac{\epsilon}{b-a} \right]$$

Now

$g_n \longrightarrow g$ uniformly in $[a,b]$, hence given $\epsilon > 0$, there exists N such that for all $n \geq N$

$$|g_n - g| < \frac{\epsilon}{b-a} \implies g_n - \frac{\epsilon}{b-a} < g < g_n + \frac{\epsilon}{b-a}.$$

So

$$\int_a^b g_n(x) - \epsilon = L(g_n - \frac{\epsilon}{b-a}) \leq L(g)$$

and

$$U(g) \leq U(g_n + \frac{\epsilon}{b-a}) = \int_a^b g_n(x) + \epsilon.$$

Hence

$$0 \leq U(g) - L(g) \leq 2\epsilon$$

implies $U(f) = L(f)$ hence f is integrable.

c) Let $Q = \{r_1, r_2, \dots\}$ be an enumeration.
Let us define $h_n : \mathbb{R} \longrightarrow \mathbb{R}$ as

$$h_n(x) = \begin{cases} 1/n & \text{if } x \in \{r_{n+1}, r_{n+2}, \dots\} \\ 0 & \text{otherwise} \end{cases}$$

h_n is not integrable for every $n \in \mathbb{N}$, $|h_n| \leq \frac{1}{n}$ for every n,
hence

$$f_n \longrightarrow f = 0$$

uniformly.
$f(x) = 0$ is an integrable function.

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Exercise 2 (7.3.6)

Let $\{r_1, r_2, r_3, \dots\}$ be an enumeration of all the rationals in $[0,1]$, and define

$$g_n(x) = \begin{cases} 1 & \text{if } x = r_n \\ 0 & \text{otherwise} \end{cases}$$

a) Is $G(x) = \sum_{n=1}^{\infty} g_n(x)$ integrable on $[0,1]$?

b) Is $F(x) = \sum_{n=1}^{\infty} \frac{g_n(x)}{n}$ integrable on $[0,1]$?

Proof.

a)

$$G(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise} \end{cases}$$

This function is not integrable since for any partition P, of $[0,1]$ any closed interval contains a rational and an irrational number.

Hence

$U(G,P) = 1; L(G,P) = 0$ hence

$$U(G) = 1 \neq 0 = L(G),$$

hence G is not integrable.

b) Consider the partial sums

$$s_n(x) = \sum_{k=1}^{n} \frac{g_k(x)}{n} = \begin{cases} 1/l & \text{if } x = r_l \in \mathbb{Q} \cap [0,1]; 1 \leq l \leq n \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 1/n & \text{if } x = r_n \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Note that $s_n \longrightarrow F$ uniformly since

$$|s_n(x) - F(x)| = \begin{cases} 1/l & \text{if } x = r_l \in \mathbb{Q} \cap [0,1]; l > n \\ 0 & \text{otherwise} \end{cases}$$

hence

$$|s_n(x) - F(x)| < \frac{1}{n}$$

hence $s_n \longrightarrow F$ uniformly and each s_n has finitely many discontinuities hence is integrable. This implies $F(x)$ is integrable.

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Exercise 3 (7.3.7)

Assume $f : [a,b] \longrightarrow \mathbb{R}$ is integrable.

a) Show that if g satisfies $g(x) = f(x)$ for all but a finite number of points in $[a,b]$, then g is integrable as well.

b) Find an example to show that g may fail to be integrable if it differs from f at a countable number of points.

Proof.

a) Consider the function

$$h(x) = g(x) - f(x).$$

Say $g(x) \neq f(x)$ for a finite set $A \subset [a, b]$.

Then

$$h(x) = \begin{cases} 0 & \text{if } x \in [a, b] \backslash A \\ g(x) - f(x) & \text{if } x \in A \end{cases}$$

Then $h(x) : [a, b] \longrightarrow \mathbb{R}$ is a function with finitely many discontinuity points.

Hence $h(x)$ is integrable.

$$g(x) = h(x) + f(x)$$

and since $h(x)$ and $f(x)$ are integrable hence $g(x)$ is integrable.

b) Let

$f(x) = 0$ and

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [a, b] \\ 0 & \text{otherwise} \end{cases}$$

Here, $f(x) \neq g(x)$ when $x \in \mathbb{Q} \cap [a, b]$ which is a countable set.

f is integrable but g is not integrable.

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