MAT 127B
HW 24 Solutions(7.4.7/7.4.8/7.4.11)

Exercise 1 (7.4.7)

Review the discussion immediately preceding Theorem 7.4.4

(a) Produce an example of a sequence $f_n \longrightarrow 0$ pointwise on $[0,1]$ where $\lim_{n\to\infty}\int_0^1 f_n$ does not exist.

(b) Produce an example of a sequence g_n with $\int_0^1 g_n \longrightarrow 0$ but $g_n(x)$ does not converge to zero for any $x \in [0,1]$. To make it more interesting, let's insist that $gn(x) \geq 0$ for all x and n.

Proof.

a) Let

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ n^2 & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} \leq x \leq 1 \end{cases}$$

$$\int_0^1 f_n(x) = \int_0^{\frac{1}{n}} n^2 = n^2\frac{1}{n} = n.$$

This implies

$$\lim_{n\to\infty}\int_0^1 f_n(x) = \lim_{n\to\infty} n$$

diverges (hence do not exist).

For $x = 0$,
$0 = f_n(0) \longrightarrow 0 = f(0)$.

For any $x > 0$,
there exists $N_x > \frac{1}{x}$, such that for all $n \geq N_x$,
$\frac{1}{n} < x$.
Hence for every $n \geq N_x$,

$$0 = f_n(x) \longrightarrow 0 = f(x).$$

Hence $f_n \longrightarrow 0$ pointwise.

b) Let us define

$$m_n = \frac{n(n+1)}{2}$$

Every $n \in \mathbb{N}$ can be written as a pair (k, l) where

$$n = m_{k-1} + l$$

where $1 \leq l \leq k$.
$n_1 = (k_1, l_1), n_2 = (k_2, l_2)$, then if $n_1 < n_2$ if $k_1 < k_2$ or if $k_1 = k_2$ then $l_1 < l_2$.
Define:

$$f_{(k,l)}(x) = \begin{cases} 1 & \text{if } x = \{\frac{t}{k} | 0 \leq t \leq k\} \\ 1 & \text{if } \frac{l-1}{k} < x < \frac{l}{k} \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^1 f_{(k,l)}(x) = \int_{\frac{l-1}{k}}^{\frac{l}{k}} 1 = \frac{1}{k}.$$

This implies

$$\lim_{n\to\infty} \int_0^1 f_n(x) = \lim_{k\to\infty} \frac{1}{k} = 0.$$

Given $x \in [0,1] \cap \mathbb{Q}$, then

$$x = p/q = np/nq$$

for every $n \in \mathbb{N}$ $f_{(nq,l)}(x) = 1$ for every n, l.
Hence $f_n(x) \nrightarrow 0$
For every $x \in \mathbb{Q}^c \cap [0,1]$, and for any given k, there exists an l_k such that $\frac{l_k-1}{k} < x < \frac{l_k}{k}$.
Hence, $f_{k,l_k}(x) = 1$ for every k, hence $f_n(x) \nrightarrow 0$.
Hence f_n does not converge to 0 but the $\int_0^1 f_n(x)$ does not converge to 0.

∎

Exercise 2 (7.4.8)

For each $n \in \mathbb{N}$, let

$$h_n(x) = \begin{cases} 1/2^n & \text{if } 1/2^n < x \leq 1 \\ 0 & \text{if } 0 \leq x \leq 1/2^n \end{cases}$$

and set $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Show H is integrable and compute $\int_0^1 H$

Proof.
$H(x)$ is an increasing function.
For $x \geq y$ for any $n \in \mathbb{N}$, $h_n(x) \geq h_n(y)$, hence

$$H(x) = \lim_{k\to\infty} \sum_{n=1}^{k} h_n(x) \geq \lim_{k\to\infty} \sum_{n=1}^{k} h_n(y) = H(y).$$

Since $H(x)$ is an increasing function, $H(x)$ is integrable.

Taken any partition
$P = \{0 = x_0, x_1, \dots, x_n = 1\}$, and if f is increasing, then

$$U(f,P) = \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}); \qquad L(f,P) = \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}).$$

Hence

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})][(x_k - x_{k-1})]$$

Consider the sequence of partitions P_n where $P_n = \{0, \frac{1}{n}, \dots, \frac{k}{n}, \dots, 1\}$.
Hence

$$\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = \lim_{n\to\infty} \frac{1}{n}[f(1) - f(0)] = 0.$$

This is equivalent as f is integrable.

$$H(x) = \begin{cases} \frac{1}{2^{n-1}} & \text{if } \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}}; \quad n \geq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Hence the integral or the area under the curve:

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \left(\frac{1}{2^{n-1}} - \frac{1}{2^n} \right) = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \left(\frac{1}{2^n} \right) = 2 \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{2}{3}.$$

∎

Exercise 3 (7.4.11)

Review the original definition of integrability in Section 7.2, and in particular the definition of the upper integral $U(f)$. One reasonable suggestion might be to bypass the complications introduced in Definition 7.2.7 and simply define the integral to be the value of $U(f)$. Then every bounded function is integrable! Although tempting, proceeding in this way has some significant drawbacks. Show by example that several of the properties in Theorem 7.4.2 no longer hold if we replace our current definition of integrability with the proposal that $\int_a^b f(x) = U(f)$ for every bounded function f.

Proof.
Assume f and g are integrable, in the new sense,

$$\int_0^1 f = U(f).$$

a)

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \in \mathbb{Q}^c \cap [0,1] \end{cases}$$

Then $\int_0^1 f(x) = 1$.

$$g(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 1 & \text{if } x \in \mathbb{Q}^c \cap [0,1] \end{cases}$$

Then $\int_0^1 g(x) = 1$.

$(f+g)(x) = 1$.

Hence $\int_0^1 f + g = 1$.

Hence

$$\int_0^1 f + \int_0^1 g > \int_0^1 f + g.$$

b) If f is integrable, then we have $|f|$ may not be integrable.

Let $\mathbb{Q} \cap [0,1]$ have an enumeration.

$\mathbb{Q} \cap [0,1] = \{r_1, r_2, \ldots, r_n, \ldots,\}$

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^c \cap [0,1] \\ -i & \text{if } x = r_i; \quad i = \{1, 2, \ldots,\} \end{cases}$$

$U(f) = 0$, since every interval has an irrational number, hence for any partition P,

$$U(f, P) = 0 \implies U(f) = 0.$$

$$|f|(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^c \cap [0,1] \\ i & \text{if } x = r_i; \quad i = \{1, 2, \ldots, \} \end{cases}$$

Taken any partition P and taken any interval, $[x, y]$ there exists infinitely many rationals, hence there exists a sequence of rationals according to the enumeration, hence

$$U(|f|) \longrightarrow \infty. \text{ (diverges) },$$

hence $|f|$ is not integrable.

■