8.4.9

(a) Show that the improper integral $\int_a^\infty f$ converges if and only if, for all $\epsilon > 0$ there exists $M > a$ such that whenever $d > c \geq M$ it follows that

$$\left| \int_c^d f \right| < \epsilon.$$

(In one direction it will be useful to consider the sequence $a_n = \int_a^{a+n} f$.)

We show the forward direction first. Fix $\epsilon > 0$. Because $\int_a^\infty f$ converges, say to some $C \in \mathbb{R}$, there exists $M > a$ such that $|\int_a^b f - C| < \frac{\epsilon}{2}$ for all $b > M$. It follows that for any $d > c \geq M$, we have $|\int_c^d f| = |\int_a^d f - \int_a^c f| \leq |\int_a^b f - C| + |\int_a^c -C| < \epsilon$. Now we show the backward direction. Consider the sequence $\{a_n\}_{n=1}^\infty := \{\int_a^{a+n} f\}_{n=1}^\infty$. We know from the assumption that there exists some integer $M > a$ such that $|a_n - a_m| = |\int_a^{a+n} f - \int_a^{a+m} f| = |\int_{a+m}^{a+n} f| < \epsilon$ for any $m, n > M - a$. So the sequence is Cauchy, which means it converges, say to some limit C'. We than have for any $b > M$ that $|\int_a^b f - C'| \leq |\int_a^b f - \int_a^{\lfloor b \rfloor} f| + |\int_a^{\lfloor b \rfloor} f - C'| < 2\epsilon$, where $\lfloor b \rfloor$ is the greatest integer that is less than or equal to b. This completes the proof. □

(b) Show that if $0 \leq f \leq g$ and $\int_a^\infty g$ converges then $\int_a^\infty f$ converges.

We know from the assumption and the monotonicity of integrals that f and g are both integrable on all $[a, b] \subseteq \mathbb{R}$ with $0 \leq \int_a^b f \leq \int_a^b g$. Fix $\epsilon > 0$, because $\int_a^\infty g$ converges, with part (a) we can find $M > a$ such that $|\int_c^d g| < \epsilon$ for any $d > c \geq M$. This means that $|\int_c^d f| < \epsilon$ for any $d > c \geq M$, completing the proof. □

(c) Part (a) is a Cauchy criterion, and part (b) is a comparision test. State and prove an absolute convergene test for improper integrals.

Claim: if $\int_a^\infty |f|$ converges, then $\int_a^\infty f$ converges.

Proof. Observe that $-|f| \leq f \leq |f|$, so with the monotonicity of integrals $-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$ for any $[a, b]$. With a similar argument as in part (b) we can show f satisfies the Cauchy criterion for improper integrals. □

8.4.10

(a) Use the properties of e^t previously discussed to show

$$\int_0^\infty e^{-t} dt = 1.$$

We know from previous disccusions that $e^x \to 0$ as $x \to -\infty$ (for a stronger result see Exercise 8.4.5). For any $[a, b] \subseteq \mathbb{R}$, we have $\int_a^b e^{-t} dt = e^{-a} - e^{-b} < e^{-a} \to 0$ as $a \to \infty$. This shows that $\int_a^\infty e^{-t} dt$ converges. Let $a = 0$, we get $\int_0^\infty e^{-t} dt = \lim_{b\to\infty} \int_0^b e^{-t} dt = 1 - \lim_{b\to\infty} e^{-b} = 1$ □

(b) Show

$$\frac{1}{\alpha} = \int_0^\infty e^{-\alpha t} dt, \quad \text{for all } \alpha > 0$$

Fix $\alpha > 0$. With a similar argument as above we know $\int_0^\infty e^{-\alpha t} dt$ converges. It follows that $\int_0^\infty e^{-\alpha t} dt = \lim_{b\to\infty} \int_0^b e^{-\alpha t} dt = \frac{1}{\alpha} - \frac{1}{\alpha} \lim_{b\to\infty} e^{-\alpha b} = \frac{1}{\alpha}$ □