MAT 127B
HW 26 Solutions(8.3.9/8.3.10/8.3.11)

Exercise 1 (8.3.9)

Theorem 8.3.1

(Integral Remainder Theorem).
Let f be differentiable $N+1$ times on $(-R,R)$ and assume $f^{(N+1)}$ is continuous. Define $a_n = f^{(n)}(0)/n!$ for $n = 0,1,\ldots,N$, and let

$$S_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N.$$

For all $x \in (-R,R)$, the error function $E_N(x) = f(x) - S_N(x)$ satisfies

$$E_N(x) = \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N dt.$$

Proof.
The case $x = 0$ is easy to check, so let's take $x \neq 0$ in $(-R,R)$ and keep in mind that x is a fixed constant in what follows. To avoid a few technical distractions, let's just consider the case $x > 0$.

a) Show

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

b) Now use a previous result from this section to show

$$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x-t)dt.$$

c) Continue in this fashion to complete the proof of the theorem.

Proof.

a) From the fundamental Theorem of Calculus,

We have $f : [0, x] \longrightarrow \mathbb{R}$ is integrable. and

$$\int_0^x f'(t) = f(x) - f(0).$$

Hence

$$f(x) = f(0) + \int_0^x f'(t).$$

b) Let $g(t) = x - t : [0, x] \longrightarrow \mathbb{R}$.

Consider the function

$$(f'(t)g(t))' = f''(t)g(t) + f'(t)g'(t).$$

From fundamental Theorem of Calculus.

$$\int_0^x (f'(t)g(t))' = \int_0^x f''(t)g(t) + \int_0^x f'(t)g'(t).$$

$$\int_0^x f''(t)g(t) = \int_0^x (f'(t)g(t))' - \int_0^x f'(t)g'(t) = f'(x)g(x) - f'(0)g(0) - \int_0^x f'(t)g'(t)$$

$$= f'(x)(0) - f'(0)(x) - \int_0^x f'(t)(-1) = -f'(0)x + f(x) - f(0).$$

$$f(x) = f(0) + f'(0)x + \int_0^x f''(t)(x - t).$$

c) Let us assume that the theorem be true for $k = N$.

Let us prove the theorem for $k = N + 1$.

Let f be $N + 2$ differentiable function on $(-R, R)$ with $f^{(N+2)}$is continuous.

This implies f is an $N+1$ differentiable function on $(-R, R)$ with $f^{(N+1)}$ is continuous.

Hence, we have

$$E_N(x) = f(x) - S_N(x)$$

holds true.

Define $a_n = f^{(n)}(0)/n!$ for $n = 0, 1, \ldots, N$, and let

$$S_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N.$$

For all $x \in (-R, R)$, the error function $E_N(x) = f(x) - S_N(x)$ satisfies

$$E_N(x) = \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N dt.$$

$$\frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N dt$$

$$= \frac{1}{N!}\left[f^{(N+1)}(x)\frac{-(x-x)^{N+1}}{N+1} - f^{(N+1)}(0)\frac{-(x-0)^{N+1}}{N+1}\right] - \frac{1}{N!}\int_0^x f^{(N+2)}(t)\frac{-(x-t)^{N+1}}{N+1}dt$$

$$= \frac{1}{(N+1)}!f^{(N+1)}(0)x^{N+1} + \frac{1}{(N+1)!}\int_0^x f^{(N+2)}(t)(x-t)^{N+1}dt$$

Define

$$a_{N+1} = f^{(N+1)}(0)/(N+1)!.$$

Hence we have

$$E_N(x) = a_{N+1}x^{N+1} + E_{N+1}(x).$$

We have,

$$E_{N+1}(x) + a_{N+1}x^{N+1} = E_N(x) = f(x) - S_N(x).$$

Hence

$$E_{N+1}(x) = f(x) - [S_N(x) + a_{N+1}x^{N+1}] = f(x) - S_{N+1}(x).$$

This proves the induction hypothesis.

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Hence the theorem is proved

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Exercise 2 (8.3.10)

a) Make a rough sketch of $1/\sqrt{1-x}$ and $S_2(x)$ over the interval $(-1, 1)$, and compute $E_2(x)$ for $x = 1/2, 3/4$, and $8/9$.

b) For a general x satisfying $-1 < x < 1$, show

$$E_2(x) = \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}}dt.$$

c) Explain why the inequality

$$\left|\frac{x-t}{1-t}\right| \leq |x|.$$

is valid, and use this to find an overestimate for $|E_2(x)|$ that no longer involves an integral. Note that this estimate will necessarily depend on x. Confirm that things are going well by checking that this overestimate is in fact larger than $|E_2(x)|$ at the three computed values from part (a).

d) Finally, show $E_N(x) \longrightarrow 0$ as $N \longrightarrow \infty$ for an arbitrary $x \in (-1, 1)$.

Proof.

a) From the previous Exercise 1:

We have:

$$E_2(x) = f(x) - S_2(x).$$

$$S_2(x) = a_0 + a_1 x + a_2 x^2,$$

where $a_i = f^{(i)}(0)/i!$.

$$f^{(0)}(0) = 1 = a_0; \qquad f^{(1)}(0) = 1/2 = a_1; \qquad f^{(2)}(0)/2! = 3/8 = a_2$$

Hence

- $x = 1/2$, we have

$$f(1/2) = \sqrt{2}; \qquad S_2\left(\frac{1}{2}\right) = \frac{43}{32}; \qquad E_2\left(\frac{1}{2}\right) = \sqrt{2} - \frac{43}{32} \approx 0.0704.$$

- $x = 3/4$, we have

$$f(3/4) = 2; \qquad S_2\left(\frac{3}{4}\right) = \frac{203}{128}; \qquad E_2\left(\frac{3}{4}\right) = 2 - \frac{203}{128} \approx 0.414.$$

- $x = 8/9$, we have

$$f(8/9) = 3; \qquad S_2\left(\frac{8}{9}\right) = \frac{47}{27}; \qquad E_2\left(\frac{8}{9}\right) = 3 - \frac{47}{27} \approx 1.2592.$$

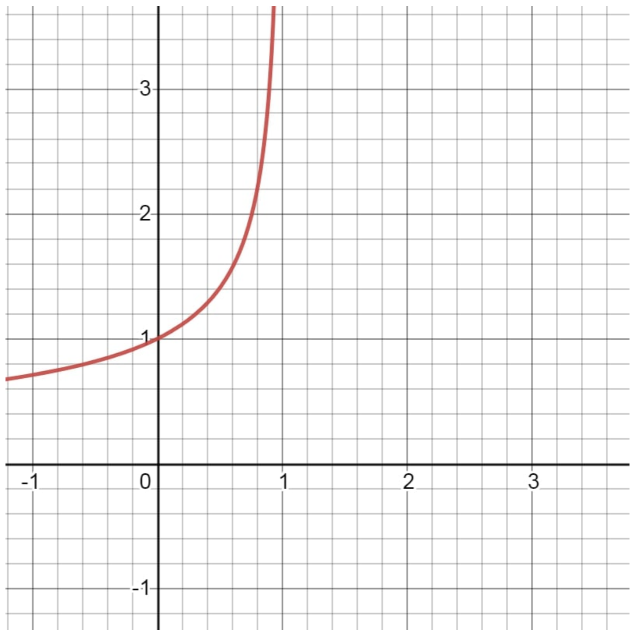

Figure 1: Graph of f(x)

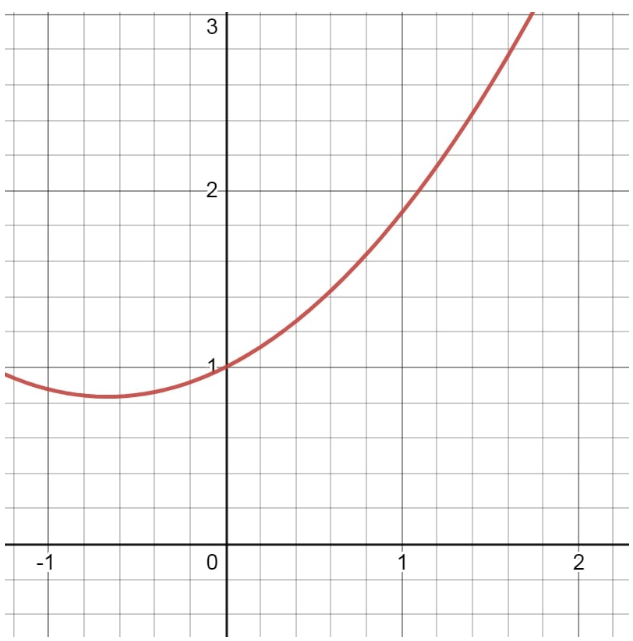

Figure 2: $S_2(x)$

b) For a given $f(x)$ which is twice differentiable and $f^{(2)}$ is continuous in $(-1,1)$, then

$$E_2(x) = \frac{1}{2!}\int_0^x f^3(t)(x-t)^2 dt.$$

$$f^{(0)}(x) = \frac{1}{\sqrt{1-x}}; \quad f^{(1)}(x) = \frac{1}{2}\frac{1}{(1-x)^{3/2}}; \quad f^{(2)}(x) = \frac{3}{4}\frac{1}{(1-x)^{5/2}};$$

$$f^{(3)}(x) = \frac{15}{8}\frac{1}{(1-x)^{7/2}}$$

$$E_2(x) = \frac{1}{2!}\int_0^x \frac{15}{8}\frac{1}{(1-t)^{7/2}}(x-t)^2 dt = \frac{15}{16}\int_0^x \frac{1}{(1-t)^{3/2}}\left(\frac{(x-t)}{(1-t)}\right)^2 dt$$

c) We have either of the two inequalities that hold for x, t

$$-1 < 0 \le t \le x < 1; \qquad -1 < x \le t \le 0 < 1.$$

Hence for the 1st inequality, we have

$$\begin{aligned} & x < 1 \\ \Longrightarrow\ & tx \le t \\ \Longrightarrow\ & x - tx = x(1-t) \ge x - t > 0 \text{ and } 1 - t > 0 \\ \Longrightarrow\ & |x| = x \ge \frac{x-t}{1-t} = \left|\frac{x-t}{1-t}\right|. \end{aligned}$$

Hence for the 2nd inequality, we have

$$\begin{aligned} & x < 1 \text{ and } t \le 0 \\ \Longrightarrow\ & tx \ge t \\ \Longrightarrow\ & x - tx = x(1-t) \le x - t < 0 \text{ and } 1 - t > 0 \\ \Longrightarrow\ & x \le \frac{x-t}{1-t} < 0 \\ \Longrightarrow\ & |x| = -x \ge \frac{-(x-t)}{1-t} = \left|\frac{x-t}{1-t}\right|. \end{aligned}$$

$$E_2(x) = \frac{15}{16}\int_0^x \left(\frac{x-t}{1-t}\right)^2 \frac{1}{(1-t)^{3/2}} dt \le \frac{15}{16}\int_0^x (|x|)^2 \frac{1}{(1-t)^{3/2}} dt$$

$$= \frac{15|x|^2}{16}\int_0^x \frac{1}{(1-t)^{3/2}}dt = \frac{15|x|^2}{16}2\left[\frac{1}{(1-t)^{1/2}}\right]\Bigg|_0^x = \frac{15x^2}{8}\left[\frac{1}{(1-x)^{1/2}}\right] - \frac{15x^2}{8} = A_2(x)[let]$$

- $$E_2(1/2) \approx 0.0704 < 0.1 < 0.15 < 0.1942 \approx A_2(1/2)$$
- $$E_2(3/4) \approx 0.414 < 0.5 < 0.8 < 1.0547 \approx A_2(3/4)$$
- $$E_2(8/9) \approx 1.2592 < 2 < 2.5 < 2.963 \approx A_2(8/9).$$

d)

$$f^{(0)}(x) = \frac{1}{\sqrt{1-x}}$$

$$f^{(n)}(x) = \prod_{k=1}^{n}(2k-1)\frac{1}{2^n}\frac{1}{(1-x)^{(2n+1)/2}} \qquad n \geq 1.$$

Proof by induction:

We already know this is true for $n = 1$.

Say it is true for $k = n$.

Then

$$f^{(n+1)}(x) = \left(f^{(n)}\right)' = \prod_{k=1}^{n}(2k-1)\frac{1}{2^n}\left(\frac{1}{(1-x)^{(2n+1)/2}}\right)' = \prod_{k=1}^{n}(2k-1)\frac{1}{2^n}\frac{2n+1}{2}\frac{1}{(1-x)^{(2n+3)/2}}$$

$$f^{(n+1)}(x) = \prod_{k=1}^{n}(2k-1)\frac{1}{2^n}\frac{2n+1}{2}\frac{1}{(1-x)^{(2n+3)/2}} = \prod_{k=1}^{n+1}(2k-1)\frac{1}{2^{n+1}}\frac{1}{(1-x)^{(2n+3)/2}}.$$

Hence

$$E_N(x) = \frac{1}{N!}\int_0^x f^{(N+1)}(t)(x-t)^N dt = \frac{1}{N!}\int_0^x \prod_{k=1}^{N+1}(2k-1)\frac{1}{2^{N+1}}\frac{1}{(1-t)^{(2N+3)/2}}(x-t)^N dt.$$

$$= \frac{1}{N!}\prod_{k=1}^{N+1}(2k-1)\frac{1}{2^{N+1}}\int_0^x \frac{1}{(1-t)^{N+(3/2)}}(x-t)^N dt$$

$$= \frac{1}{N!}\prod_{k=1}^{N+1}(2k-1)\frac{1}{2^{N+1}}\int_0^x \frac{1}{(1-t)^{(3/2)}}\left(\frac{x-t}{1-t}\right)^N dt$$

Hence,

$$|E_N(x)| \leq \frac{1}{N!}\prod_{k=1}^{N+1}(2k-1)\frac{1}{2^N}|x|^N\left[\frac{1}{\sqrt{(1-x)}}-1\right]$$

Since $|x|<1$, hence $\exists k$ such that $|x|^k < \frac{1}{2}$. (since $|x|^k \longrightarrow 0$.)

Given $|x|<1$, consider the sequence

$$s_N = \frac{1}{N!}\prod_{k=1}^{N+1}(2k-1)\frac{|x|^N}{2^N} = \prod_{k=1}^{N}\left[\frac{(2k+1)|x|}{2k}\right]$$

$$|E_n(x)| \leq s_N\left[\frac{1}{\sqrt{(1-x)}}-1\right]$$

and if $s_n \longrightarrow 0$, then

$$E_N \longrightarrow 0 \text{ as } N \longrightarrow \infty.$$

Hence we want to prove that

$s_n \longrightarrow 0$.

We will show that given $|x|<1$ there exists $N_x \in \mathbb{N}$ such that s_n is decreasing for $n \geq N_x$.

Moreover we know that

$$s_n \geq 0 \qquad \forall n \in \mathbb{N}.$$

This implies $s_n \longrightarrow a \geq 0$.

We will show

$$s_n \longrightarrow 0$$

This proves the theorem.

Given $|x| < 1$,

consider the sequence

$$a_N = \frac{2N}{2N+1}.$$

This sequence is increasing (since $a_N = 1/(1+1/2n)$, $2n$ is increasing, hence $1+3/2n$ is decreasing, $1/(1+3/2n)$ is increasing), and converges to 1.

This proves there exists N_x,

such that for $n > N$

$$|x| < \frac{2n}{2n+1}.$$

Consider for $n > N_x$

$$\frac{s_n}{s_{n-1}} = \frac{(2n+1)|x|}{2n} < 1.$$

Hence for $n > N_x$

$$\{s_n\} \text{ is a decreasing sequence. }.$$

Consider

$$\frac{(2n+1)|x|}{2n}.$$

This sequence is decreasing and converges to $|x| < 1$.

Fix an r such that $|x| < r < 1$.

Then there exists $N_x^{(2)}$ such that for all $n > N_x^{(2)}$.

$$\frac{(2n+1)|x|}{2n} < r$$

Take $N = \max\{N_x, N_x^{(2)}\}$

Now, for $n > N$

$$s_n < \frac{(2n+1)|x|}{2n} s_{n-1} < r s_{n-1}.$$

Hence for $n > N$, we have

$$s_n < r s_{n-1} < r^2 s_{n-2} < \ldots r^{n-N} s_N = r^n \frac{s_N}{r^N}.$$

Hence

$$s_n < r^n \frac{s_N}{r^N} \implies \lim_{n\to\infty} s_n \leq \lim_{n\to\infty} r^n \frac{s_N}{r^N} = 0.$$

This implies $s_n \longrightarrow 0$ and hence the theorem is proved.

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Exercise 3 (8.3.11)

Assuming that the derivative of $\arcsin(x)$ is indeed $1/\sqrt{1-x^2}$
supply the justification that allows us to conclude

$$\arcsin(x) = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1} \text{ for all } |x| < 1.$$

Proof.
We have that

$$\sqrt{1-x^2} = \sum_{n=0}^{\infty} c_n x^{2n}$$

for $|x| < 1$.
Since this is a power series which converges uniformly for $|x| < 1$, hence $R(\sqrt{1-x^2}) \geq 1$.
From the fundamental theorem of calculus since the function is continuous in $(-1, 1)$ hence $\sqrt{1-x^2}$ is integrable and

$$G(x) = \int_0^x \sqrt{1-x^2} dt$$

is a differentiable function on $(-1, 1)$ with $G'(x) = \sqrt{1-x^2}$. and $R(G(x)) \geq 1$.
and

$$G(x) = \int_0^x \sqrt{1-t^2} dt = \int_0^x \sum_{n=0}^{\infty} c_n t^{2n} dt = \sum_{n=0}^{\infty} \int_0^x c_n t^{2n} dt = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} [t^{2n+1}]|_0^x$$

For $G'(x) = (\arcsin x)' = \sqrt{1-x^2}$, $G(0) = \arcsin(0) = 0$,
hence $G(x) = \arcsin x$.

$$\arcsin x = \sum_{n=0}^{\infty} \frac{c_n}{2n+1} x^{2n+1}.$$

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