MAT 127B-A

Winter 2021
Left Board
Newton 1666 undergrad at Cambridge while away from school due to plague.

Computation from MAT 21.

Analogous space: plane or $\mathbb{R}^2$

Nice pts

Rational pts $(\frac{a}{b}, \frac{c}{d})$

Approx $(\pi, \sqrt{3}) \approx (3.1415\ldots, 3.1622\ldots)$
(3.3), (3.1, 3.1), ... 

Both are examples of Banach Algebras.

(Ch 13).

(Many more examples in Functional Analysis course).
Def: 8.1 If \( a < c < b \) and \( f: (a, b) \to \mathbb{R} \) then write \( f'(c) = (Df)(c) = \lim_{h \to 0} \frac{f(h+c)-f(c)}{h} = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \).

If \( f'(c) \) exists say \( f \) is differentiable.
at $c$.

If $f$ is diff. at every value in $(a, b)$

say $f$ is diff. in or on $(a, b)$.

Compute $f'_3(x)$ using chain and

product rules away from $x=0$
and the design at $x=0$

$$f'_3(x) = \begin{cases} 
2x \sin \frac{1}{x} + x^2 \left( \frac{-1}{x^2} \cos \frac{1}{x} \right) & x \neq 0 \\
\end{cases}$$

$$f'_3(0) = \lim_{h \to 0} \frac{f_3(h)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h}$$

$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

Squeeze or sandwich this.
Try in breakout rm:

\[ g_1(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \quad \text{odd} \]

\[ g_2(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases} \quad \text{even} \]

\( \text{where are neg cts.} \)

\( \text{where different.} \)
\[ g'(0) = \lim_{h \to 0} \frac{g_1(h)}{h} \]

\[-h \leq g_1(h) \leq h\]

\[-1 \leq \frac{g_1(h)}{h} \leq 1\]

Sandwich Theorem fails.

If \( \lim_{h \to 0} \frac{h \sin \frac{1}{h}}{n} \) exists.

\[ = \lim_{n \to 0} \sin \frac{1}{n} \]

\[ = \lim_{h \to 0} \sin \frac{1}{h} \]

\[ = \lim_{h \to 0} \sin \frac{1}{x_i} = 1 \]

\[ \sin \frac{1}{x_i} = \sin \frac{1}{\frac{(2i+1)\pi}{2}} \]

\[ \sin \frac{1}{x_i} = \sin \frac{1}{\frac{1}{2}} = 1 \]

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If \( g_i = \frac{1}{(2i+\frac{1}{2})\pi} \)
\[
\text{ten } \sin \left( \frac{1}{\gamma_i} \right) = \sin \left( (2i - \frac{1}{2}) \pi \right) = -1
\]

and \( \lim_{n \to 0} \sin \frac{1}{n} = -1 \)
Def: 6.1 If \( f: (a, b) \to \mathbb{R} \) and \( a < c < b \), then \( \lim_{x \to c} f(x) = L \) if

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in (a, b) \text{ with } |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon.
\]

Def: If \( f: (a, b) \to \mathbb{R} \) and \( a < c < b \) then

1. \( f \) is \underline{continuous (cts)} at \( c \) if \( \lim_{x \to c} f(x) = f(c) \).
2. \( f \) is \underline{differentiable (diff)} at \( c \) if \( f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \) exists.
Section 8.18

**Theorem:**

If $f$ is continuously differentiable (ctdy diff) at $c$, if $f(x)$ is diff. in an interval around $c$ and $f'$ is cts at $c$.

**Notation**

Work required to check the examples:

1. Check: $|x| \text{ is cts.}$
   - $\text{at } x \neq 0$
   - $\text{at } x = 0$
   - $|x| \text{ is not diff at } x = 0$

   Need $\lim_{x \to c} f(x) = f(c)$ use def. of limit.
Need \( \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x} \) does not exist.

Use the definition.

2. Check: \( f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{at } x = 0 \end{cases} \) is differentiable from last time.

\[ f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{at } x = 0 \end{cases} \]

Choose \( x = 0 \) and show:

\[ \lim_{x \to 0} f'(x) \neq f'(0) = 0 \]
Assume for contradiction that \( \lim_{x \to 0} f'(x) = 0 \).

Hence: \( \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \) with \( |x| < \delta \) it is true that \( |f'(x)| < \frac{\varepsilon}{3} \).

Choose \( \varepsilon = \frac{1}{3} \).

Hence by the hypothesis there is a \( \delta > 0 \) and choose \( x = \frac{1}{2\pi n} \) with \( n \) big enough that \( x < \delta \).

Now compute: \( \exists \xi > \frac{1}{2} > |f'(\xi)| = |2x \sin \frac{1}{x} - \cos \frac{1}{x}| \).
\[ = \left| \frac{1}{2\pi n} \sin(2\pi n) - \cos(2\pi n) \right| \]
\[ = \left| 0 - 1 \right| = 1 \]

which is a contradiction.

Hence \( \lim_{x \to 0} f'(x) \neq 0 = f'(0) \)

so \( f'(x) \) is discontinuous at \( 0 \).

so \( f(x) \notin C^1 \mathbb{R} \).
Thm 8.17: If \( f'(c) \) exists then \( f \) is cts at \( c \).

Proof:

Recall properties of limits:

If \( \lim_{x \to a} g(x) = L \) and \( \lim_{x \to a} k(x) = M \)

then

\[
\lim_{x \to a} (g(x) + k(x)) = L + M
\]

\[
\lim_{x \to a} (g(x) \cdot k(x)) = L \cdot M
\]
Assume: $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

want to show $\lim_{x \to c} f(x) = f(c)$.

Compute: $\lim_{x \to c} (x - c) = 0$

$$f(c) = f(c) + \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c)$$

$$= \lim_{x \to c} \left[ f(c) + \frac{f(x) - f(c)}{x - c} \cdot x - c \right]$$

$$= \lim_{x \to c} f(x)$$

done.
$C^k$ means $f^{(k)}$ exists and is $k^{th}$ deriv. differentiable.

$D^k$ means $f^{(k)}$ exists.

$C^\infty$ means for every $k$, $f^{(k)}$ is continuous. 

$D^\infty$ means for every $k$, $f^{(k)}$ exists.

E.g. $\sin(x)$ is $f^{(1)} = \cos(x)$, $f^{(2)} = -\sin(x)$, ... $f^{(k)}$.
Algebraic operations on functions and how they relate to 127A: Limits/Continuity.

Today: Derivatives.

\[ \text{Fun}(\mathbb{R}) \supseteq C^0(\mathbb{R}) \supseteq D'(\mathbb{R}) \supseteq C'(\mathbb{R}) \supseteq D^2(\mathbb{R}) \ldots \]

all functions \( f: \mathbb{R} \to \mathbb{R} \)
**Thm 8.19/8.21**: If $f$ and $g$ are diff. on $\mathbb{R}$, then:

$$(kf)'(c) = k(f'(c))$$

$$(f+g)'(c) = f'(c) + g'(c)$$

$$(g \cdot f)'(c) = g'(c)f(c) + g(c)f'(c)$$

$$(f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)} \quad (\text{if } g(c) \neq 0)$$

$$(D(kf)) = k(Df)$$

$$D(f+g) = Df + Dg$$

$$D(g \cdot f) = (Dg) \cdot f + g(Df)$$

$$D\left(\frac{f}{g}\right) = \frac{(Df) \cdot g - f \cdot (Dg)}{g^2}$$
\[(gof)'(a) = \[(g'of)(c)] \cdot f'(a)\]

Hence:

If \( f, g \in D'(\mathbb{R}) \) then \( k \cdot f, f + g, f \circ g \) \( \in D'(\mathbb{R}) \) by the above rules.

If \( f, g \in C^1(\mathbb{R}) \) so \( f', g' \in C^0(\mathbb{R}) \) (both cts.)

\[
D^2(\mathbb{R}) = C^2(\mathbb{R}) = C^0(\mathbb{R})
\]
Proof of chain rule:
Consider $g, f \in D'(\mathbb{R})$.

Write $G_a(x) = \begin{cases} \frac{g(x)-g(a)}{x-a} & x \neq a \\ g'(a) & x = a \end{cases}$

which is cts at $a$ iff $g$ is diff at $a$. 
Also \( f \in D'(\mathbb{R}) \) is diff. so facts

So by above:

\[
\lim_{x \to c} (G \circ f)(x) = \lim_{y \to f(c)} G(y).
\]

Take \( a = f(c) \).

Now compute:

\[
(g \circ f)'(c) = \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c}.
\]

Def. \( x \to c \)

\[
= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \frac{f(x) - f(c)}{x - c}
\]

\[
= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)}.
\]
\[
\begin{align*}
&= \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \cdot \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \\
&= \lim_{y \to f(c)} \frac{g(y) - g(f(c))}{y - f(c)} \\
&= g'(f(c)) \cdot f'(c)
\end{align*}
\]
To show $D^\infty(R) = C^\infty(R)$

Def: $f \in D^\infty(R)$ if for every $k$, $f^{(k)}$ exists.

Hence if $f \in D^\infty(R)$, then $f^{(k+1)}$ exists.

So $f^{(k)}$ is diff.

So $f^{(k)}$ is cts.

So $f \in C^\infty(R)$