MAT 127 B-A
Winter 2021
Right Board
Overview:
Define (again) Derivative
Integral
Study spaces of functions
- consider nice functions
  - poly nominals
  - trig polynomials
- consider limits of these to get more functions
\[ e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \]

useful as a soln to a diff eqn

common in phys.

Weierstrass fn:

\[ w(x) = \cos(x) + \frac{1}{2} \cos(3x) + \frac{1}{4} \cos(7x) + \cdots \]

extremely jagged.

Similar functions come up in physics as paths of particles in a field.
Example:

1. \( f(x) = x^2 \)
   \[
   f'(3) = 2 \cdot 3 = 6 \quad \text{(rule)} \quad (x^2)' = 2x
   \]

or using def:

\[
\begin{align*}
   f'(3) &= \lim_{h \to 0} \frac{f(h+3) - f(3)}{h} \\
   &= \lim_{h \to 0} \frac{(h+3)^2 - 3^2}{h} \\
   &= \lim_{h \to 0} \frac{h^2 + 6h + 9 - 9}{h} \\
   &= \lim_{h \to 0} \frac{h^2 + 6h}{h} \\
   &= \lim_{h \to 0} (h + 6) = 6
\end{align*}
\]
Compute \( f'_2(x) = \begin{cases} 2x & x > 0 \\ 0 & x = 0 \\ 0 & x < 0 \end{cases} \)

Since derivatives are local rules work in intervals

But at \( x = 0 \) use the definition

\[
f'_2(0) = \lim_{h \to 0} \frac{f_2(h+0) - f_2(0)}{h} = \lim_{h \to 0} \frac{f_2(h)}{h}
\]
Recall: \( \lim_{x \to a} F(x) = L \quad \text{iff} \quad \lim_{x \to a^+} F(x) = L = \lim_{x \to a^-} F(x) \).

Here:
\[
\lim_{h \to 0^+} \frac{f_2(h)}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = \lim_{h \to 0^+} h = 0
\]
\[
\lim_{h \to 0^-} \frac{f_2(h)}{h} = \lim_{h \to 0^-} \frac{0}{h} = \lim_{h \to 0^-} 0 = 0
\]
g.

cts at 0
not diff at 0 since:

$$g_2' (0) = \lim_{h \to 0} \frac{g_2(h)}{h}$$
use squeeze thm:

\[ 0 \leq g_2(h) \leq 2h^2 \]

so \[ 0 \leq \frac{g_2(h)}{h} \leq h \]

\[ \lim_{h \to 0} = 0 \leq \lim_{h \to 0} \frac{g_2(h)}{h} \leq \lim_{h \to 0} h = 0 \]

\[ \checkmark \]
Notation:
If $f$ is cts in $(a,b)$ write $f \in C([a,b])$
If $f$ is diff in $(a,b)$ write $f \in D'(a,b)$
If $f$ is ctsly diff in $(a,b)$ write $f \in C'([a,b])$

Picture: $C^0([a,b]) \supset D'(a,b) \supset C'(a,b) \supset D^2(a,b)$...
Containment requires proof.
Proper content (not equal) requires examples.

Example 1: \( f(x) = |x| \) has \( f(x) \in C^0 \mathbb{R} \)
but \( f(x) \notin D'(\mathbb{R}) \)

Example 2: \( f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \) has \( f(x) \in D' \mathbb{R} \)
but \( f(x) \notin C^1 \mathbb{R} \)

In breakout RMS:

1. More examples to see
\[ c^0 \neq D^1 \]
\[ D^1 \neq c^1 \]
\[ c^1 \neq D^2 \] New

<table>
<thead>
<tr>
<th>( c^0 ) not ( D^1 )</th>
<th>( D^1 ) not ( c^1 )</th>
<th>( c^1 ) not ( D^2 )</th>
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<tbody>
<tr>
<td>( x^3 \sin \frac{1}{x} )</td>
<td>( \begin{cases} x^2 &amp; \text{if } x &gt; 0 \ 0 &amp; \text{if } x \leq 0 \end{cases} )</td>
<td>( \frac{1}{x^2} )</td>
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What are \( c^\infty \) and \( D^\infty \)?

<table>
<thead>
<tr>
<th>Same ( c^\infty ) ( D^\infty )</th>
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<tbody>
<tr>
<td>( \exp ) ( \text{trig} ) ( \frac{1}{x^2} )</td>
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where does $x^3 \sin \frac{1}{x}$ live?
If \( f, g : \mathbb{R} \to \mathbb{R} \) and \( k \in \mathbb{R} \)

**Linearity:** \( k \cdot f \in \text{Fun}(\mathbb{R}) \)

\[ f + g \in \text{Fun}(\mathbb{R}) \]

**Product:** \( f \cdot g \in \text{Fun}(\mathbb{R}) \)

**Composition:** \( g \circ f \in \text{Fun}(\mathbb{R}) \)

(or \( f \cdot g \in \text{Fun}([0,1]) \))

eg:

\[ f = \frac{1}{x}, \quad g = \sin(x) \]

\[ k = 3 \]

\[ 3 \cdot \frac{1}{x} = k \cdot f \]

\[ \sin(x) + \frac{1}{x} = f \cdot g \]

\[ \frac{1}{x} \sin(x) = f \circ g \]

\[ g \circ f = \sin \left( \frac{1}{x} \right) \]

\[ f \circ g = \frac{1}{\sin(x)} \]
Continuity: If $f, g$ cts (or $f, g \in C^0(\mathbb{R})$), then $rof, f+g, fog, gof$ are also cts.

In particular if $G$ is cts at $f(c)$ and $f$ is cts at $c$, then

$$
\lim_{x \to c} (G \circ f)(x) = (G \circ f)(c) = G(f(c)) = \lim_{y \to f(c)} G(y)
$$

G of cts at $c$

**Eg:** $\lim_{x \to \pi} \sin\left(\frac{1}{x}\right) = \sin \frac{1}{\pi} = \lim_{y \to \frac{1}{\pi}} \sin(y)$
\[
(3 \sin(x))' = 3 \sin'(x) = 3 \cos(x)
\]
\[
(sin(x) + \frac{1}{x})' = \cos(x) - \frac{1}{x^2}
\]
\[
(\frac{1}{x} \sin(x))' = -\frac{1}{x^2} \sin(x) + \frac{1}{x} \cos(x)
\]
\[
\left(\frac{\sin(x)}{x}\right)' = \frac{x \cos(x) - 1 \cdot \sin(x)}{x^2}
\]
\[
[\sin(\frac{1}{x})]' = \cos(\frac{1}{x}) \cdot (-\frac{1}{x^2})
\]
Proof of Chain Rule:

Use

Parametric Derivs:

Claim: \( \lim_{s \to t} \frac{y(s) - y(t)}{x(s) - x(t)} = \frac{y'(t)}{x'(t)} \)

and this is also the slope of the tangent line.

(use for L'Hôpital rule if \( x'(t) \neq 0 \))
What are possible target line slopes at $x'(t) = 0$? Typically the line is vertical.

but any value is possible e.g. if $y' = x' = 0$