1. Spectra F14.2

Fall 2014 Problem 2: Define the bounded linear operator $K : L^2(0,1) \to L^2(0,1)$ by

$$(Kf)(x) = \int_0^1 xy(1-xy)f(y)dy.$$

Find the spectrum or K and classify it.

1.1. Ideas F14.2.

- K is self-adjoint.
- K has a tiny (at most two dimensional) image.
- Operator direct sums work well with spectra.

1.2. Write up F14.2.

Lemma: If $F \in CI^2$ and K_F is the bounded linear integral operator on $L^2 I$ then $K_F^* = K_{\overline{F}}$.

From the lemma, since $xy(1-xy)$ is real valued on I^2 and symmetric one has $K^* = K$.

Lemma: If $T^* = T$ then $T = T|_{\Im(T)} \oplus T|_{Ker(T)}$.

The image of K is contained in $\langle x, x^2 \rangle$ since $Kf = x\langle y, f(y)\rangle - x^2\langle y^2, f(y)\rangle$. Thus the kernel is infinite dimensional and the spectrum is all point spectrum and is 0 and the eigenvalues of the 2 by 2 matrix of $K|_{\langle x,x^2\rangle}$.

Computing ...$\{0, \frac{4\pm\sqrt{31}}{60}\}$.

2. Spectra F13.5

Fall 2013 Problem 5: Suppose that A is a compact operator on an infinite dimensional Hilbert space $\mathcal{H}$. Show that A does not have a bounded inverse operator.

2.1. Ideas F13.5.

- If $A = A^*$ the spectrum contains 0 (the only element of the spectrum that might not be an eigenvalue).
- The topological definition of a compact operator that the image of the unit ball is precompact is a strong restriction if there is a continuous inverse.
- The unit ball itself is not precompact.

2.2. Write up F13.5.

Assume for contradiction that A has a bounded inverse. If A is a bounded linear operator with a bounded inverse then A is continuous with continuous inverse or equivalently a homeomorphism. If A is compact then the image of the unit ball is precompact.

Lemma: The unit ball of an infinite dimensional Hilbert space is not precompact.

Homeomorphisms preserve precompactness, giving a contradiction.