1. Fourier Series F18.3

Fall 2018 Problem 3: Show that for every $f \in C(\mathbb{T})$ and $\epsilon > 0$ there is an initial condition $g \in C(\mathbb{T})$ for which there is a solution $u(x,t)$ to the heat equation on a ring with $u(x,0) = g(x)$ and $|u(x,1) - f(x)| < \epsilon$ for every $x \in \mathbb{T}$.

1.1. Ideas F18.3.

- Recall the heat equation is $u_t = u_{xx}$.
- Is the maximum principle helpful?
- Try a Fourier series in $x \in \mathbb{T}$.
- The heat equation is smoothing so this would be false for $\epsilon = 0$.
- Stone-Weierstrass gives smooth trigonometric approximations to try for $u(\cdot, 1)$.

2. Fourier Series F14.6

Fall 2014 Problem 6:

(a) By choosing a suitable even, periodic extension for f, calculate the Fourier cosine series for $f = \sin x$, $x \in [0, \pi]$.

(b) Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = 1/2.$$

2.1. Ideas F14.6.

- Sketch f and extend to $f(x) = |\sin(x)|$ on $[-\pi, \pi]$.
- Integrate by writing in terms of exponentials.
- Integrate by parts (twice?).
- Remember or work out normalization with an example.

3. Fourier Transform S16.4

Spring 2016 Problem 4: Suppose f is a function in the Schwartz space $S(\mathbb{R})$ which satisfies the normalizing condition $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$. Let $\hat{f}$ denote the Fourier transform of f. Show that

$$\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx\right) \left(\int_{-\infty}^{\infty} \omega^2 |\hat{f}(\omega)|^2 d\omega\right) \geq \frac{1}{16\pi^2}.$$

3.1. Ideas S16.4.

- This is Heisenberg uncertainty (normalization?).
- The normalization is $f \in L^2\mathbb{R}$.
- The LHS is $\|xf\|_2^2 \|w\hat{f}\|_2^2$.
- $w\hat{f} = \pm i\widehat{f'}$.

- $\|f\|_2\|g\|_2 \geq \|fg\|_1$ (Hölder).
- $\|f\|_2 = \|\hat{f}\|_2$ (Parseval).
- Schwartz will eliminate the boundary terms in integration by parts.