1. Fourier Series F18.4

Fall 2018 Problem 4: Consider the functions $f_N(x) = (2\pi)^{-1}\sum_{n=-N}^{N} e^{inx}$. Show that if $g \in L^2(\mathbb{T})$ then $\{f_N * g\}$ converges in $\|\cdot\|_{L^2}$-norm to g (here, $*$ denotes convolution).

1.1. Ideas F18.4.

- f_N is given as a Fourier series.
- The Fourier series is an isometry.
- Convolution works well with Fourier series, turning into pointwise multiplication.
- Convergence in $\|\cdot\|_{L^2}$-norm can arise from an onb expansion.

1.2. Write up F18.4.

Compute

(1) $\widehat{f_N * g}$
(2) $= \widehat{f_N} \cdot \hat{g}$
(3) $= \chi_{|n|\le N} \cdot \hat{g}$
(4) $= \sum_{|n|\le N} \langle \chi_{\{n\}}, \hat{g}\rangle \chi_{\{n\}}$
(5) which converges in $\|\cdot\|_{\ell^2}$-norm to $\hat{g}$.
(6) Thus $\{f_N * g\}_N$ converges in $\|\cdot\|_{L^2}$-norm to g.

Justifications::

(1)-(2) **Theorem:** If $a, b \in L^2\mathbb{T}$ then $\widehat{a * b} = \hat{a} \cdot \hat{b}$ in $\ell^2\mathbb{Z}$.
(2)-(3) $\widehat{e^{inx}} = 2\pi\chi_{\{n\}}$.
(3)-(4) Rewriting in terms of the ℓ^2 inner product.
(5) **Theorem:** If $\{x_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for a (separable) Hilbert space H and $y \in H$ then $\{\sum_{|n|\le N}\langle x_n, y\rangle x_n\}_N$ converges in norm to y.
(6) **Theorem:** $\hat{\cdot} : L^2\mathbb{T} \to \ell^2\mathbb{Z}$ is an isometry of Hilbert spaces and hence preserves norm convergence of sequences.

2. Fourier Series F16.2

Fall 2016 Problem 2: Let $f : \mathbb{T} \to \mathbb{C}$ be a C^1 function such that $\int_{-\pi}^{\pi} f(x)dx = 0$. Show that

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \leq \int_{-\pi}^{\pi} |f'(x)|^2 dx.$$

2.1. **Ideas F16.2.**

- $\mathbb{T}$ suggests Fourier series.
- $\int |f|^2$ is $\| \cdot \|_{L^2}^2$ which also suggests Fourier.
- $\widehat{f'} = \pm in\hat{f}$ suggests Fourier.
- $\int f = \hat{f}(0)$ suggests Fourier.

2.2. **Write up F16.2.** Compute

(1) $\int_{-\pi}^{\pi} |f'(x)|^2 dx$
(2) $= \|f'\|_{L^2}^2$
(3) $= \|\widehat{f'}\|_{\ell^2}^2$
(4) $= \|n\hat{f}(n)\|_{\ell^2}^2$
(5) $\geq \hat{f}^2(0) + \|\hat{f}\|_{\ell^2}^2$
(6) $= \int_{-\pi}^{\pi} f(x)dx + \|f\|_{L^2}^2$
(7) $= 0 + \int_{-\pi}^{\pi} |f(x)|^2 dx.$

Justifications:

(1)-(2) This is the defintition.
(2)-(3) **Theorem:** If $a \in L^2\mathbb{T}$ then $\|a\|_{L^2} = \|\hat{a}\|_{\ell^2}$.
(3)-(4) **Theorem:** If $a \in C^1\mathbb{T} \subseteq L^2\mathbb{T}$ then $\widehat{a'}(n) = n\hat{a}(n)$.
(4)-(5) Expanding and using $n^2 \geq 1$ unless $n = 0$.
(5)-(6) This is the definition and another use of the isometry from (3).
(6)-(7) This is the hypothesis and definition of the norm.