1. Hilbert F13.1

Fall 2013 Problem 1:

Find $\inf_f \int_0^1 |f(x)-x|^2 dx$ where the infimum is taken over all $f \in L^2([0,1])$ such that $\int_0^1 f(x)(x^2-1)dx = 1$.

1.1. Ideas F13.1. :

- Consider an orthogonal projection.
- Work with $g(x) = f(x) - x$ instead of $f(x)$.
- Thus one is trying to minimize the norm of g subject to a given inner product so the minimum should occur with g a multiple of $x^2 - 1$.

2. Hilbert S16.6

Spring 2016 Problem 6:

Let H be a Hilbert space and let U be a unitary operator, that is surjective and isometric, on H. Let $I = \{v \in H : Uv = v\}$ be the subspace of invariant vectors with respect to U.

(1) Show that $\{Uw - w : w \in H\}$ is dense in $I^\perp$ and that I is closed.

(2) Let P be the orthogonal projection onto I. Show that

$$\frac{1}{N}\sum_{n=1}^{N} U^n v \to Pv.$$

2.1. Ideas S16.6.

- Rewrite I as the kernel of $U - 1$ making it closed.
- Call J the image of $U-1$ and check the containment $J \subseteq I^\perp$ by using the linearity properties of the inner product and that $(Uv, Uu) = (v, u)$.
- Consider the spectrum of $U - I$ in a unit circle.
- Check for the projection properties $P = P^2 = P^*$, or $P|_J = 0$ and $P|_I = 1$, or $\ker(P) = \operatorname{im}(P)^\perp$.
- Try to use (1): Write $S_N = \frac{1}{N}\sum_{n+1}^{N} U^n$ and compute that $Sv = v$ if $v \in I$.
- Try to use (1): Compute $S_N(U-1)w = N^{-1}(U^{N+1}w - w)$ which approaches 0.
- Check norms to see that $\|S_N\|_{op}$ is uniformly bounded and hence (1) implies (2).
- To see that J is dense in $I^\perp$ choose $x \in J^\perp \cap I^\perp = (I+J)^\perp$.
- If $x \in J^\perp$ then for every v there is $0 = (x, (U-1)v) = ((1-U)x, Uv)$ but U is onto so $(1-U)x = 0$.