1. Hilbert S14.4

Spring 2014 Problem 4:

Let P_1 and P_2 be a pair of orthogonal projections onto H_1 and H_2, respectively, where H_1 and H_2 are closed subspaces of a Hilbert space H. Prove that P_1P_2 is an orthogonal projection if and only if P_1 and P_2 commute. In that case, prove that P_1P_2 is the orthogonal projection onto $H_1 \cap H_2$.

1.1. Ideas S14.4.

- Sketch an example in $\mathbb{R}^2$.
- Recall P is an orthogonal projection iff $P = P^2 = P^*$ iff $\ker(P) = \mathrm{im}(P)^\perp$.

1.2. Write up S14.4.

(1) If: Assume $P_1P_2 = P_2P_1$.
   (a) $(P_1P_2)^* = (P_2P_1)^* = P_1^*P_2^* = P_1P_2$.
   (b) $(P_1P_2)^2 = P_1P_2P_1P_2 = P_1P_1P_2P_2 = P_1P_2$.

(2) Only if: Assume $(P_1P_2)^* = P_1P_2$.
   (a) $P_1P_2 = (P_1P_2)^* = P_2^*P_1^* = P_2P_1$.

(3) Image: $\mathrm{im}(P_1P_2) \subseteq H_1 \cap H_2$ Note that $\mathrm{im}(PQ) \subseteq \mathrm{im}(P)$ so that if $PQ = QP$ then $\mathrm{im}(PQ) = \mathrm{im}(QP) \subseteq \mathrm{im}(P) \cap \mathrm{im}(Q)$.

(4) Image: $\mathrm{im}(P_1P_2) \supseteq H_1 \cap H_2$ Note that if P is a projection and $x \in \mathrm{im}(P)$ then $x = Px$. Thus if P and Q are projections and $x \in \mathrm{im}(P) \cap \mathrm{im}(Q)$ then $x = Px = PQx \in \mathrm{im}(PQ)$.

2. Hilbert F17.6

Fall 2017 Problem 6: Consider a linear functional $\phi(f) = f(\frac{1}{2})$ defined on the space of polynomials on $[0, 1]$. Does ϕ extend to a bounded linear functional on $L^2([0, 1])$? Prove or disprove.

2.1. Ideas F17.6.

- Extension suggests Hahn-Banach might give the extension.
- Bounded suggests continuous. Recall the definition.
- Polynomials suggest (Stone)-Weierstrass might give counterexamples.
- Since L^2 functions are not defined at points there is no obvious guess for an extension so maybe look for a counterexample.
- Try to get polynomials with large value at $\frac{1}{2}$ but small 2-norm.

2.2. Write up F17.6. :

Definition: A linear functional

$$g : L^2([0, 1]) \to \mathbb{R}$$

is *bounded* if $\|g\|_{op} = \sup_{f \in L^2([0,1])-\{0\}} \frac{|g(f)|}{\|f\|_2}$ is finite.

Assume there is an extension g and compute

$$\|g\|_{op} \geq \sup_f \frac{|f(\frac{1}{2})|}{\|f\|_2}$$

with f a nonzero polynomial.

Theorem:(Weierstrass) Polynomials are uniformly dense in $C[0, 1]$.

Note that if $f \in C([0, 1])$ then $\|f\|_2 \leq \|f\|_\infty$.

For each ϵ choose $g \in C([0, 1]$ with $g(\frac{1}{2}) = 1$ and $\|g\|_2 \leq \epsilon$ and by the theorem choose $f \in \mathbb{R}[x]$ with $\|f - g\|_\infty \leq \epsilon$. Thus $|f(\frac{1}{2})| \geq 1 - \epsilon$ and $\|f - g\|_2 \leq \epsilon$. Thus $\|f\|_2 \leq \|f - g\|_2 + \|g\|_2 \leq 2\epsilon$ and $\|g\|_{op} \geq \frac{1-\epsilon}{2\epsilon}$ which is not bounded.