1. Spectra S16.2

Spring 2016 Problem 2: Let M be a multiplication on $L^2(\mathbb{R})$ defined by

$$Mf(x) = m(x)f(x),$$

where $m(x)$ is continuous and bounded. Prove that M is a bounded operator on $L^2(\mathbb{R})$ and that the spectrum is given by

$$\{m(x) : x \in \mathbb{R}\}^{cl},$$

where A^{cl} denotes the closure of A. Can M have eigenvalues?

1.1. **Ideas S16.2.**

- M is self-adjoint since the adjoint $M_m^* = M_{\overline{m}}$ another multiplication operator.
- The shift $M_m - \lambda I = M_{m-\lambda}$ is also another multiplication operator.
- The inverse $M_m^{-1} = M_{m^{-1}}$ is too if m stays bounded away from 0.
- The norm is $\|M_m\|_{op} = \|m\|_\infty$.
- If m is a constant function c then $M_m = cI$ which has one eigenvalue.

2. Spectra F12.3

Fall 2012 Problem 3: Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Show that

(a) If $\|T\| \leq 1$, then T and its adjoint operator T^* have the same fixed point. i.e. Show that for $x \in \mathcal{H}$,

$$Tx = x \Longleftrightarrow T^*x = x.$$

(b) Let λ be an eigenvalue of T. Is it true that its complex conjugate $\overline{\lambda}$ must be an eigenvalue of T^*? Is it true that $\overline{\lambda}$ must be in the spectrum of T^*? Justify your answers.

2.1. **Ideas F12.3.**

(b) Adjunction behaves well with spectra:

(bi) If λ is an eigenvalue for T (point spectrum) so that $T - \lambda$ has kernel then $T^* - \overline{\lambda}$ has cokernel (residual or point spectrum)

(bii) The image of $T - \lambda$ is dense iff that of $T^* - \overline{\lambda}$ is.

(a) Adjoints preserve operator norm. If $\|T\| = 1$ fixed points witness the norm of T.

3. Spectra W08.4

Winter 2008 Problem 4: Suppose that $A : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on a (complex) Hilbert space $\mathcal{H}$ with spectrum $\sigma(A) \subset \mathbb{C}$ and resolvent set $\rho(A) = \mathbb{C} \backslash \sigma(A)$. For $\mu \in \rho(A)$, let

$$R(\mu, A) = (\mu I - A)^{-1}$$

denote the resolvent operator of A.

(a) If $\mu \in \rho(A)$ and

$$|\nu - \mu| < \frac{1}{\|R(\mu, A)\|},$$

prove that $\nu \in \rho(A)$ and

$$R(\nu, A) = [I - (\mu - \nu)R(\mu, A)]^{-1} R(\mu, A).$$

(b) If $\mu \in \rho(A)$, prove that

$$\|R(\mu, A)\|| \geq \frac{1}{d(\mu, \sigma(A))}$$

where

$$d(\mu, \sigma(A)) = \inf_{\lambda \in \sigma(A)} |\mu - \lambda|$$

is the distance of μ from the spectrum of A.

3.1. Ideas W08.4.

- Try a power series expansion for $[I - (\mu - \nu)R(\mu, A)]^{-1}$.