Spring 2014 Problem 4:

Let P_1 and P_2 be a pair of orthogonal projections onto H_1 and H_2, respectively, where H_1 and H_2 are closed subspaces of a Hilbert space H. Prove that P_1P_2 is an orthogonal projection if and only if P_1 and P_2 commute. In that case, prove that P_1P_2 is the orthogonal projection onto $H_1 \cap H_2$.

Fall 2017 Problem 6: Consider a linear functional $\phi(f) = f(\frac{1}{2})$ defined on the space of polynomials on $[0, 1]$. Does ϕ extend to a bounded linear functional on $L^2([0, 1])$? Prove or disprove.

Fall 2013 Problem 1:

Find $\inf_f \int_0^1 |f(x) - x|^2 dx$ where the infimum is taken over all $f \in L^2([0,1])$ such that $\int_0^1 f(x)(x^2 - 1)dx = 1$.

Spring 2016 Problem 6:

Let H be a Hilbert space and let U be a unitary operator, that is surjective and isometric, on H. Let $I = \{v \in H : Uv = v\}$ be the subspace of invariant vectors with respect to U.

(1) Show that $\{Uw - w : w \in H\}$ is dense in $I^{\perp}$ and that I is closed.

(2) Let P be the orthogonal projection onto I. Show that

$$\frac{1}{N}\sum_{n=1}^{N} U^n v \to Pv.$$

Fall 2016 Problem 3:
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real valued scalars and assume that the series $\sum_{n=1}^{\infty} a_n x_n$ converges for every $x \in \ell^2(\mathbb{N})$. Show that $y \in \ell^2(\mathbb{N})$.

Fall 2016 Problem 6:
Let D denote the closed unit disk in $\mathbb{C}$, and consider the complex Hilbert space

$$\mathcal{H} := \left\{ f : D \to \mathbb{C} | f(z) = \sum_{k=1}^{\infty} a_k z^k \text{ and } \|f\|_{\mathcal{H}}^2 := \sum_{k=0}^{\infty} (1+k^2)|a_n|^2 < \infty \right\}.$$

Prove that the linear functional $L : D \to \mathbb{C}$ defined by $L(f) = f(1)$ is bounded, and find an element $g \in \mathcal{H}$ such that $L(f) = \langle g, f \rangle$. (In other words so that g represents L as in the Riesz representation theorem.)