Fall 2018 Problem 2: Consider the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} x\log x & \text{if } x\in(0,1]\\ 0 & \text{if } x=0.\end{cases}$$

Be sure to justify your answer for each of the following questions.

(a) Is f Lipschitz continuous on $[0,1]$?
(b) Is f uniformly continuous on $[0,1]$?
(c) Suppose (p_n) is a sequence of polynomial functions on $[0,1]$, converging uniformly to f. Is the set $A=\{p_n|n\geq 1\}\cup\{f\}$ equicontinuous?

(Spring 2015 Problem 1.) Let $\mathcal{H}$ be a separable Hilbert space. A sequence (x_n) in $\mathcal{H}$ *converges in the Cesáro sense* to $x\in\mathcal{H}$ if the averages of its partial sums converge strongly to x, i.e., if

$$\overline{x}_N=\frac{1}{N}\sum_{n=1}^{N}x_n\to x \quad \text{as } N\to\infty.$$

(a) Prove that if (x_n) converges strongly to x in $\mathcal{H}$, then (x_n) also converges in the Cesáro sense to x.
(b) Give an example of a sequence that converges in the Cesáro sense but does not converge weakly.
(c) Give an example of a sequence that converges weakly but does not converge in the Cesáro sense.

(Spring 2015 Problem 4) Let $f_n:[0,1]\to\mathbb{R}$ be a sequence of measurable functions. Suppose

(i) $\int_0^1|f_n(x)|^2dx\leq 1$ for $n=1,2,\cdots$.
(ii) $f_n\to 0$ almost everywhere.

Show that

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx=0.$$

(Fall 2015 Problem 1.) Consider the following sequences of functions parameterized by n:

- $a_n(x)=e^{i2\sqrt{n}\pi x},\ \ x\in[0,1]$,
- $b_n(x)=\sqrt{n}e^{-n|x|},\ \ x\in\mathbb{R}$,
- $c_n(x)=ne^{-nx^2},\ \ x\in\mathbb{R}$,
- $d_n(x)=\sum_{k=-n}^{n}e^{i2k\pi x},\ \ x\in[0,1]$.

As n tends to infinity, which sequences converge (a) almost everywhere, (b) L^2-strongly, (c) L^2-weakly but not strongly. Explain your answer.

(Fall 2014 Problem 3) Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle : H\times H\to\mathbb{C}$, and let $\{e_n : n\in\mathbb{N}\}$ be an orthonormal basis of H. Define a metric $d: B\times B\to\mathbb{R}$ on the closed unit ball B of H by

$$d(x,y)=\sum_{n=1}^{\infty}\frac{|\langle x-y,e_n\rangle|}{2^n}.$$

(a) Show that the sequence (x_k) in B converges to $x\in B$ with respect to the metric d if and only if it converges weakly to x in H.

(b) Prove that (B, d) is a compact metric space.

(Spring 2014 Problem 3) Let $f, f_k : E \to [0, +\infty)$ be nonnegative Lebesgue integrable functions on a measurable set $E \subseteq \mathbb{R}^n$. If (f_k) converges to f pointwise almost everywhere and

$$\int_E f_k dx \to \int_E f dx$$

show that

$$\int_E |f - f_k| dx \to 0.$$