Winter 2008 Problem 5: Let $1 \leq p < \infty$ and let $I = (-1, 1)$ denote the open interval in $\mathbb{R}$. Find the values of α as a function of p for which the function $|x|^\alpha \in W^{1,p}(I)$.

Fall 2009 Problem 1: For $\epsilon > 0$, let η_ϵ denote the family of *standard* mollifiers on $\mathbb{R}^2$. Given $u \in L^2(\mathbb{R}^2)$, define the functions

$$u_\epsilon = \eta_\epsilon \star u \text{ in } \mathbb{R}^2.$$

Prove that

$$C\|Du_\epsilon\|_{L^2(\mathbb{R}^2)} \leq \|u\|_{L^2(\mathbb{R}^2)},$$

where the constant C depends on the mollifying function but not on u.

Fall 2012 Problem 4: The heat kernel on $\mathbb{R}^3$ is given by $H_t(x) = (4\pi t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4t}}$ where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^3$. Prove that if $u \in L^3(\mathbb{R}^3)$, then $t^{\frac{1}{2}} \|H_t * u\|_{L^\infty(\mathbb{R}^3)} \to 0$ as $t \to 0^+$. (Note that $*$ denotex convolution.)

Fall 2013 Problem 3: For $\delta > 0$ small, let $u \in L^{\frac{3}{2}+\delta}(\mathbb{R}^3) \cap L^{\frac{3}{2}-\delta}(\mathbb{R}^3)$. Prove that $v = u * \frac{1}{|x|} \in L^\infty(\mathbb{R}^3)$ and provide a bound for $\|v\|_{L^\infty(\mathbb{R}^3)}$ which depends only on $\|u\|_{L^{\frac{3}{2}+\delta}(\mathbb{R}^3)}$.

Spring 2010 Problem 5: Let $H^s(\mathbb{R})$ denote the Sobolev space of order s on the real line $\mathbb{R}$, and let

$$\|u\|_s = \left(\int_{\mathbb{R}} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

denote the norm on $H^s(\mathbb{R})$ where $\hat{u}(\xi) = \frac{1}{2\pi}\int u(x)e^{-ix\xi}dx$ denotes the Fourier transform of u.
Suppose that $r < s < t$, all real, and $\epsilon > 0$ is given. Show that there exists a constant $C > 0$ such that

$$\|u\|_s \leq \epsilon\|u\|_t + C\|u\|_r \qquad \forall u \in H^t(\mathbb{R}).$$

Spring 2013 Problem 6: Let I be the interval $(0,1)$ and $q \geq p \geq 1$. Show that there exists a constant $C = C(p,q,I)$ such that

$$\|u\|_{L^q(I)} \leq C\|u\|_{W^{1,p}(I)}$$

for all $u \in W_0^{1,p}(I)$. (Hint: First show that $\|u\|_{L^\infty(I)} \leq C\|u\|_{W^{1,p}(I)}$ for all $u \in W_0^{1,p}(I)$.)

Fall 2015 Problem 4: Let $f : \mathbb{R}^3 \to \mathbb{R}$ with $f, \nabla f \in L^1(\mathbb{R}^3)$. Show that

$$\int_{\mathbb{R}^3} |f(x)|^{\frac{3}{2}} dx \leq \left(\int_{\mathbb{R}^3} |\nabla f(x)| dx \right)^{\frac{3}{2}} .$$