Problem 6, Fall 2018: Let $\Omega = \{(x,y) : y \geq 0, x \in \mathbb{R}\}$. Let $f \in C_c^1(\mathbb{R}^2)$ (space of continuous functions with compact support and with continuous first derivatives). Show the following

$$\int_{\mathbb{R}} |f(x,0)|^2 dx \leq 2 \left(\int_{\Omega} |f(x,y)|^2 dxdy + \int_{\Omega} \left| \frac{\partial f}{\partial y}(x,y) \right|^2 dxdy \right).$$

Problem 4, Fall 2017: Let $[a,b] \subseteq \mathbb{R}$ be a closed interval and let

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)| \qquad \|f\|_2 = \sqrt{\int_a^b |f(x)|^2 dx}$$

denote the L^∞ and L^2 norms. If $f \in C^1([a,b])$, prove that

$$\|f\|_\infty^2 \leq \frac{\|f\|_2^2}{b-a} + 2\|f\|_2 \|f'\|_2.$$

Fall 2014 Problem 5: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a smooth (C^∞) function with compact support. Prove that

$$\lim_{n \to \infty} \left\{ \sqrt{\frac{8}{\pi}} \int_0^\infty n \sin(n^2 x^2) f(x) dx \right\} = f(0).$$

Hint. You can use the fact that

$$\lim_{a \to \infty} \left\{ \int_0^a \sin(t^2) dt \right\} = \sqrt{\frac{\pi}{8}}.$$

Fall 2016 Problem 6: Let D denote the closed unit disk in $\mathbb{C}$, and consider the complex Hilbert space

$$\mathcal{H} = \left\{ f : D \to \mathbb{C} \,\middle|\, f(z) = \sum_{k=0}^\infty a_k z^k \text{ and } \|f\|_{\mathcal{H}}^2 = \sum_{k=0}^\infty (1+k^2)|a_k|^2 < \infty \right\}.$$

Prove that the linear functional $L : \mathcal{H} \to \mathbb{C}$ defined by $L(f) = f(1)$ is bounded, and find an element $g \in \mathcal{H}$ such that $L(f) = \langle g, f \rangle$. (In other words, so that g represents L as in the Riesz representation theorem.)

Fall 2014 Problem 1: Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable. Suppose $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all $x \in \mathbb{R}$ Prove or disprove that $|f'(x)| \leq 2$ for all $x \in \mathbb{R}$.