Spring 2014 Problem 3 Let $f, f_k : E \to [0, +\infty)$ be nonnegative Lebesgue integrable functions on a measurable set $E \subseteq \mathbb{R}^n$. If (f_k) converges to f pointwise almost everywhere and

$$\int_E f_k dx \to \int_E f dx,$$

show that

$$\int_E |f - f_k| dx \to 0.$$

Fall 2017 Problem 3 Prove or disprove the following statement: If $f \in C^\infty([0,1])$ is a smooth function, then there is sequence (p_n) of polynomials on $[0,1]$ such that $p_n^{(k)} \to f^{(k)}$ uniformly on $[0,1]$ as $n \to \infty$ for every integer $k \geq 0$. Here $f^{(k)}$ denotes the kth derivative of f.

Fall 2015 Problem 6 Suppose that $f \in S(\mathbb{R})$, where $S(\mathbb{R})$ is the Schwartz space of infinitely differentiable rapidly decreasing functions

$$S(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} |x^n f^{(m)}(x)| < \infty\}$$

for all nonnegative integers

$$n, m = 0, 1, 2, \ldots.$$

Does

$$\int_{\mathbb{R}} f(x)x^n dx = 0, \quad n = 0, 1, 2, \ldots.$$

imply that f is identically zero? Explain your answer. (Hint: use Fourier transform).

Note: $f^{(m)}$ denotes the m-th derivative of f.

Fall 2013 Problem 2 Let $L^2([0,1])$ denote the Hilbert space of complex valued square integrable functions on $[0,1]$ with the usual inner product

$$(f,g) = \int_0^1 f(x)\overline{g(x)}dx.$$

Define $T : L^2([0,1]) \to L^2([0,1])$ by

$$(Tf)(x) = \int_0^x f(t)dt, \quad \text{for } x \in [0,1].$$

(a) Show that T is bounded.
(b) Show that T has no eigenvalues.
(c) Find $\lim_{n\to\infty} \|T^n\|$.

Spring 2014 Problem 2 Let T be a linear operator from a Banach space X to a Hilbert space H. Show that T is bounded if and only if $x_n \rightharpoonup x$ implies that $T(x_n) \rightharpoonup T(x)$ for every weakly convergent sequence (x_n) in X.

Spring 2015 Problem 6 Let $\{A_n\}$ be a sequence of bounded linear operators on a Hilbert space H that converges weakly to an operator A, and suppose that for each $x \in H$ one has $\|A_n x\| \to \|Ax\|$ as $n \to \infty$. Prove that A_n strongly converges to A (in particular, for unitary operators weak convergence to a unitary operator implies strong convergence).

Fall 2016 Problem 1: Let A and B be two bounded, self-adjoint operators on a Hilbert space $\mathcal{H}$. Prove that

$$\|Af\|\|Bf\| \geq \frac{|\langle [A,B]f, f\rangle|}{2},$$

where $[A,B] = AB - BA$ is the commutator of A and B. In addition prove that equality holds if and only if $Af = cBf$ for some $c \in \mathbb{R}$.