3. Problems: 200-2015-3: Hilbert

Problem 3.1. *(Wi 02 5)(Wi 04 5)*

(1) *Show that every subset A of a Hilbert space has $(A^{\perp})^{\perp}$ the closure of its linear span.*
(2) *Show that every closed convex subset of a Hilbert space contains a unique element of minimal norm.*

Problem 3.2. *(Fa 04 5) Show that if x and y are elements of a normed linear space X so that $f(x) = f(y)$ for every $f \in X^*$ then $x = y$.*

Problem 3.3. *(Wi 06 5) Take $\{u_k\}_{k\in\mathbb{N}}$ to be an orthonormal set in a Hilbert space X. Characterize those sequences of scalars $\{a_k\}$ so that $\{a_k u_k\}$ is compact in X.*

Problem 3.4. *(Wi 09 3)*

(1) *Show that if $\{x_n\}$ is a sequence of elements of a Hilbert space $\mathbb{H}$ converging weakly to x and $\|x_n\|$ also converges to $\|x\|$ then the sequence converges to x in norm.*
(2) *Find a sequence of elements $\{x_n\}$ of a Hilbert space $\mathbb{H}$ converging weakly to x with $\liminf_{n\to\infty} \|x_n\| > \|x\|$.*

Problem 3.5. *(Fa 09 4) Find examples of sequences of bounded operators on a Hilbert space which:*

(1) *converge weakly but not strongly*
(2) *converge strongly but not in norm*
(3) *Show that every bounded operator on a seperable Hilbert space is the strong limit of a sequence of finite rank operators.*

Problem 3.6. *(Fa 10 6) Deterimine for each pair either by proof or counterexample whether a sequence of elements in a Hilbert space:*

(1) *which converges weakly must also converge strongly*
(2) *which converges strongly must also be bounded*
(3) *which converges weakly must also be bounded*
(4) *which is bounded must have a subsequence which converges strongly*
(5) *which is bounded must have a subsequence which converges weakly*
(6) *which converges weakly must have the images under T converging in $\mathbb{C}^n$ for every bounded operator from the Hilbert space to $\mathbb{C}^n$.*

Problem 3.7. *(Fa 12 1) Show that $(C[0,1], \|\cdot\|)$ with $\|f\| = sup_{x\in[0,1]}|f(x)|$ is not the normed linear space arising from a Hilbert space.*

Problem 3.8. *(Sp 14 2) Show that a linear operator T from a Banach space X to a Hilbert space H is bounded iff it takes weakly convergent sequences in X to weakly convergent sequences in H.*

3.1. **Demonstration Problems.**

Problem 3.9. *(Wi 03 2) Consider the Hilbert space $L^2[-1,1]$.*

(1) *Find the orthogonal complement to the subspace of polynomials.*
(2) *Find the orthogonal complement to the subspace of polynomials in x^2.*

Problem 3.10. *(Fa 03 4) Show that if z and y are elements of a Hilbert space X so that $\langle y, x\rangle = \langle z, x\rangle$ for every $x \in X$ then $y = z$.*