MAT 21B SPRING 2025 PRACTICE FINAL SOLUTIONS

(1) A function has the folowing values:

x	0	2	4	6	8	10	12
$f(x)$	4	3	5	6	7	6	5

Estimate the integral $\int_0^{12} f(x)dx$ using three equal length intervals and the midpoint rule.

Solution. Dividing the interval into three equal length subintervals, we get $[0, 4] \cup [4, 8] \cup [8, 12]$, each with length 4. The midpoint of these intervals are $2, 6, 10$ respectively, so we evaluate $f(2) = 3, f(6) = 6, f(10) = 6$, and the midpoint approximation is

$$\int_0^{12} f(x)dx \approx 4 \cdot 3 + 4 \cdot 6 + 4 \cdot 6 = 60.$$

□

(2) Find the number $\displaystyle\int_{-1}^{1} (x + 1)e^{x+1}dx$.

Solution. We apply IBP with $u = (x + 1), dv = e^{x+1}dx$, so $du = dx, v = e^{x+1}$. Then

$$\begin{aligned}\int_{-1}^{1} (x + 1)e^{x+1}dx &= \left[(x + 1)e^{x+1} - \int e^{x+1}dx\right]_{-1}^{1} \\ &= \left[(x + 1)e^{x+1} - e^{x+1}\right]_{-1}^{1} \\ &= xe^{x+1}|_{-1}^{1} \\ &= e^2 + 1\end{aligned}$$

Remark. It's possible that upon seeing this problem, your first thought is to do u-sub with $u = x+1$, which is totally fine; you'd end up having to do IBP either way though. □

(3) Find the number $\displaystyle\int_0^2 \frac{dx}{9 - x^2}$.

Solution. We apply the method of Partial Fractions.

$$\begin{aligned}&\int \frac{dx}{9 - x^2} = \int \frac{dx}{(3 - x)(3 + x)} = \int \left(\frac{A}{3 - x} + \frac{B}{3 + x}\right) dx \\ &\implies 1 = A(3 + x) + B(3 - x).\end{aligned}$$

Letting $x = 3$, we get $1 = A(6) + B(0) \implies A = \frac{1}{6}$, and letting $x = -3$, we get $1 = A(0) + B(6) \implies B = \frac{1}{6}$. So

$$\begin{aligned}\int_0^2 \frac{dx}{9 - x^2} &= \int_0^2 \frac{1/6}{3 - x}dx + \int_0^2 \frac{1/6}{3 + x}dx \\ &= \frac{1}{6}(-\ln(3 - x))\Big|_0^2 + \frac{1}{6}(\ln(3 + x))\Big|_0^2 \\ &= \frac{1}{6}\left[-\ln(3 - 2) + \ln(3 - 0)\right] + \frac{1}{6}\left[\ln(3 + 2) - \ln(3 + 0)\right] \\ &= \frac{1}{6}(-\ln(1) + \ln(5)) \\ &= \frac{\ln(5)}{6}.\end{aligned}$$

□

(4) Find the antiderivative with constant of integration: $\displaystyle\int \frac{dx}{9 + x^2}$.

Solution. We use trig sub with $x = 3\tan(\theta)$, $dx = 3\sec^2(\theta)d\theta$. So

$$\int \frac{dx}{9+x^2} = \int \frac{3\sec^2(\theta)d\theta}{9+9\tan^2(\theta)} = \int \frac{3\sec^2(\theta)d\theta}{9(1+\tan^2(\theta))}$$
$$= \int \frac{1}{3}\frac{\sec^2(\theta)d\theta}{\sec^2(\theta)} = \int \frac{1}{3}d\theta = \frac{\theta}{3} + C.$$

Note that $x = 3\tan(\theta) \implies \theta = \arctan(\frac{x}{3})$, hence

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C.$$

Remark. In general, when you see $\int \frac{dx}{a^2+x^2}$, you should use trig sub with $x = a\tan(\theta)$. □

(5) Determine whether the improper integral converges: $\displaystyle\int_0^\infty \frac{dx}{9+x^2}$.

Solution. Note that from the previous problem, $\int \frac{dx}{9+x^2} = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$, so

$$\int_0^\infty \frac{dx}{9+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{9+x^2} = \lim_{b\to\infty}\frac{1}{3}\arctan\left(\frac{b}{3}\right) - \frac{1}{3}\arctan\left(\frac{0}{3}\right)$$
$$= \frac{1}{3}\cdot\frac{\pi}{2} - 0 = \frac{\pi}{6}.$$

Remark. Why is $\arctan(\infty) = \frac{\pi}{2}$? Note that $\tan = \frac{\sin}{\cos}$, so tan is ∞ happens when cos is 0, and we know $\cos(\frac{\pi}{2}) = 0$. Similarly, $\arctan(0) = 0$ because $\sin(0) = 0$, and thus $\tan(0) = 0$. □

(6) Consider the curve $y = \sqrt{9+x^2}$ between $x = 0$ and $x = 4$. Write an integral for the length of the curve. You do not need to evaluate the integral.

Solution. Recall the arc length formula is $\int_a^b \sqrt{1+(f'(x))^2}dx$. Note that we have $\frac{dy}{dx} = \frac{1}{2}(9+x^2)^{-\frac{1}{2}} \cdot \frac{d}{dx}(9+x^2) = \frac{1}{2\sqrt{9+x^2}} \cdot 2x$, and hence the length of this curve from $x = 0$ to $x = 4$ is

$$\int_0^4 \sqrt{1+\frac{x^2}{9+x^2}}dx.$$

□

(7) Consider the solid obtained by revolving about the x-axis the region bounded above by the curve $y = \sqrt{9+x^2}$, below by the x-axis and between $x = 0$ and $x = 4$. Find the volume of the solid.

Solution. We apply Washer Method, with radius of the disk equal to $\sqrt{9+x^2}$. So the volume is

$$V = \int_0^4 \pi(\sqrt{9+x^2})^2dx = \pi\int_0^4 9+x^2dx$$
$$= \pi\left[9x+\frac{x^3}{3}\right]_0^4 = \pi\left(9\cdot 4+\frac{4^3}{3}\right) = \pi(36+64/3) = \frac{172\pi}{3}.$$

□

(8) Find the work in Nm required to lift an anchor which weighs $100N$ along with its $10m$ chain which weighs $100\frac{N}{m}$.

Solution. We compute the work on the anchor first. The anchor's weight is constant, so the work on the anchor is just the force times the distance, which is $100N \cdot 10m = 1000Nm$. Now, we find the work on the chain. Note that at x meters high, $10-x$ meters of the chain is left. So the force of the chain at x meters high is $F(x) = 100\frac{N}{m}\cdot(10-x)m$, and thus the work on the chain is

$$W = \int_0^{10} F(x)dx = \int_0^{10} 100(10-x)dx = \int_0^{10}(1000-100x)dx$$
$$= 1000x - 50x^2\Big|_0^{10} = 10000 - 5000 = 5000Nm.$$

Thus, the total work is $1000 + 5000 = 6000Nm$. □

The last three problems find the slopes of the curves at the origin.

(9) Find $f'(0)$ if $f(x) = \int_0^{3x} \cos(t^2)dt$. Hint: no integration required.

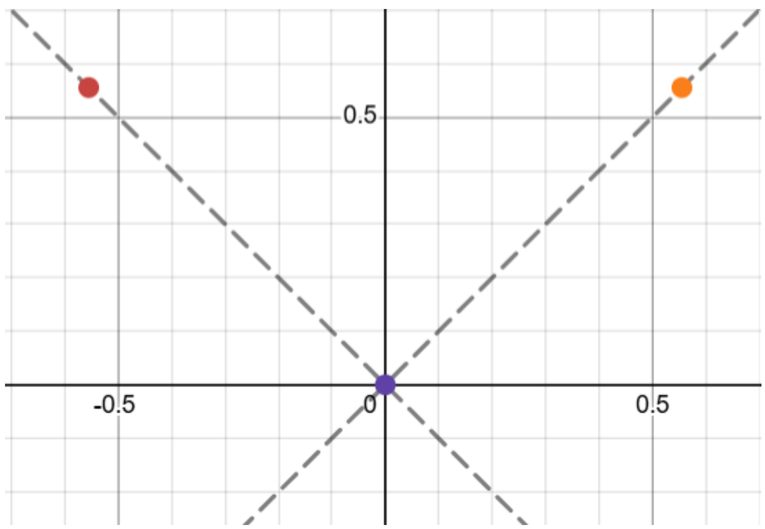
Solution. By First Fundamental Theorem of Calculus, $f'(x) = \cos((3x)^2) \cdot \frac{d}{dx}(3x) = 3\cos(9x^2)$, and thus $f'(0) = 3 \cdot \cos(0) = 3$. □

(10) Find the slope $\frac{dy}{dx}$ at the point where $t = 0$ of the curve given parametrically by $x = t + t^2$ and $y = te^t$.

Solution. Note that $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{e^t + te^t}{1 + 2t}$, and at $t = 0$, we have $\frac{dy}{dx} = \frac{1}{1} = 1$. □

(11) Consider the curve given in polar coordinates by $r = \theta$. Graph the points where θ is $\frac{-\pi}{4}$, 0 and $\frac{\pi}{4}$ and find the slope $\frac{dy}{dx}$ at the point where $\theta = 0$.

Solution. The three points in polar coordinates are $(-\frac{\pi}{4}, -\frac{\pi}{4}), (0, 0), (\frac{\pi}{4}, \frac{\pi}{4})$, and they are **red**, **purple**, and **orange** (left to right) in the following graph respectively (the dashed lines are $\theta = \frac{\pi}{4}$ and $\theta = \frac{-\pi}{4}$).



Now, we compute

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta} r\sin(\theta)}{\frac{d}{d\theta} r\cos(\theta)} = \frac{\frac{d}{d\theta}\theta(\sin(\theta))}{\frac{d}{d\theta}\theta(\cos(\theta))} = \frac{\sin(\theta) + \theta\cos(\theta)}{cos(\theta) - \theta\sin(\theta)},$$

and at $\theta = 0$, we have $\frac{dy}{dx} = \frac{\sin(0)+0}{\cos(0)-0} = \frac{0}{1} = 0$. □