MAT 21B SPRING 2025 PRACTICE MIDTERM 2 SOLUTIONS

(1) Integration

(a) $\displaystyle\int_{x=0}^{1} \frac{dx}{\sqrt{4-x^2}}$.

Solution. Let $x = 2\sin\theta$. Then $dx = 2\cos\theta d\theta$, and

$$\int \frac{1}{\sqrt{4-x^2}}dx = \int \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}d\theta = \int \frac{2\cos\theta}{2\sqrt{1-\sin^2\theta}}d\theta = \int d\theta = \theta + C$$

Note that $x = 2\sin\theta \implies \theta = \sin^{-1}\left(\frac{x}{2}\right)$, so

$$\int_{x=0}^{1} \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right)\Big|_{x=0}^{1} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) = \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

□

(b) $\displaystyle\int_{x=-1}^{1} \frac{xdx}{4-x^2}$

Solution. Consider $f(x) = \frac{x}{4-x^2}$. Note that

$$f(-x) = \frac{-x}{4-(-x)^2} = -\frac{x}{4-x^2} = -f(x),$$

so $f(x)$ is an odd function. Therefore, $\displaystyle\int_{x=-1}^{1} \frac{xdx}{4-x^2} = 0$.

You can also solve this problem using u-sub with $u = 4 - x^2$. □

(c) $\displaystyle\int_{x=-1}^{1} \frac{dx}{4-x^2}$

Solution. We use Partial Fractions:

$$\int_{x=-1}^{1} \frac{dx}{4-x^2} = \int_{-1}^{1} \frac{1}{(2-x)(2+x)}dx = \int_{-1}^{1} \left(\frac{A}{2-x} + \frac{B}{2+x}\right)dx = \int_{-1}^{1} \left(\frac{A(2+x)+B(2-x)}{(2-x)(2+x)}\right)dx$$

So we have $1 = A(2+x) + B(2-x)$. Setting $x = -2$, we get $1 = A(0) + B(4) \implies B = \frac{1}{4}$, and setting $x = 2$, we get $1 = A(4) + B(0) \implies A = \frac{1}{4}$. Thus,

$$\begin{aligned}
\int_{x=-1}^{1} \frac{dx}{4-x^2} &= \int_{-1}^{1} \frac{1/4}{2-x}dx + \int_{-1}^{1} \frac{1/4}{2+x}dx \\
&= \left(\frac{1}{4}(-\ln(2-x)) + \frac{1}{4}\ln(2+x)\right)\Big|_{-1}^{1} \\
&= \left[\frac{1}{4}(-\ln(1)) + \frac{1}{4}\ln(3)\right] - \left[\frac{1}{4}(-\ln(3)) + \frac{1}{4}\ln(1)\right] \\
&= \frac{1}{2}\ln(3).
\end{aligned}$$

□

(d) $\displaystyle\int x\ln(x)dx$

Solution. Let $u = \ln(x)$, $dv = xdx$. Then $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$, and using IBP,
$$\int x\ln(x)dx = \frac{x^2}{2}\ln(x) - \int \frac{x}{2}dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C$$
□

(e) $\displaystyle\int \frac{\sin(x)}{1-\sin^2(x)}dx$.

Solution. Note $\displaystyle\int \frac{\sin(x)}{1-\sin^2(x)}dx = \int \frac{\sin(x)}{\cos^2(x)}dx$. Let $u = \cos(x)$, $du = -\sin(x)dx$. Then
$$\int \frac{\sin(x)}{\cos^2(x)}dx = \int -\frac{1}{u^2}du = \int -u^{-2}du = -\frac{u^{-1}}{-1} = \frac{1}{u} + C = \frac{1}{\cos(x)} + C.$$
□

(f) $\displaystyle\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx$.

Solution. Let $u = \sqrt{x}$, $du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}dx$. Then $\frac{1}{\sqrt{x}}dx = 2du$, so
$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \int e^{-u}\cdot 2du = -2e^{-u} + C = -2e^{-\sqrt{x}} + C.$$
□

(2) Consider the curve $y = \sin(x)$ between $x = 0$ and $x = \frac{\pi}{2}$ (so the ends are at $(0,0)$ and $(\frac{\pi}{2}, 1)$).

(a) Write a definite integral for the length of the given curve.

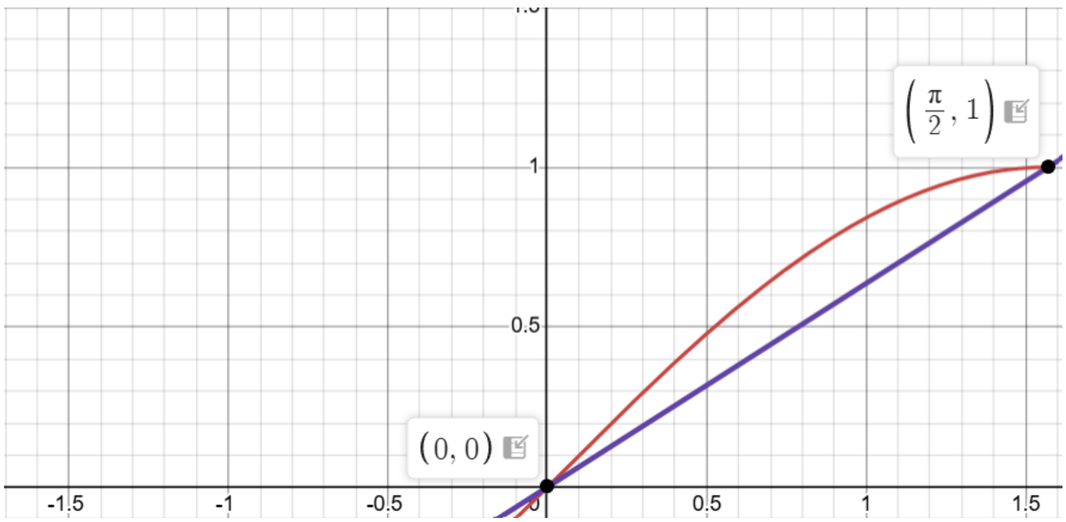
Solution. We have $f(x) = \sin(x)$, so $f'(x) = \cos(x)$. Then using the formula for arc length, we get
$$\int_0^{\frac{\pi}{2}} \sqrt{1+\cos^2(x)}dx.$$
□

(b) Write a definite integral for the area of the surface obtained by rotating the given curve about the axis $x = -2$.

Solution. Note first that we are revolving the curve about $x = -2$, which is a vertical line. So to compute the surface area obtained, we first need to rewrite $y = \sin(x)$ into $x = g(y)$. That is, $g(y) = \sin^{-1}(y)$, and this curve goes from $y = 0$ to $y = 1$. Then $g'(y) = \frac{1}{\sqrt{1-y^2}}$. Now also note that the radius (frustrum) from the curve to the axis of revolution is $x+2 = g(y)+2$, and putting the above information in the formula for surface area gives us
$$\int_0^1 2\pi(\sin^{-1}(y)+2)\sqrt{1+\frac{1}{1-y^2}}dy.$$
□

(c) Consider the region bounded above by the given curve and below by the line segment connecting its two ends. Write a definite integral for the volume of the object obtained by rotating this region about the y-axis.

Solution. We have the following region, revolved about the y-axis:



Where the straight line is $y = \frac{2}{\pi}x$. We use the shell method to get the volume to be

$$\int_0^{\frac{\pi}{2}} 2\pi x \left(\sin(x) - \frac{2}{\pi}x\right) dx$$

□