Math 21B Midterm II Spring 2025: Wed May 7 3:10-4:00

You may use one page of notes but not a calculator or textbook. There are tables of integrals and trigonometric identities on the next page. Please do not simplify your answers.

Name:	
ID:	

Basic and Trigonometric Integrals

$\int x^n dx$	$=$	$\frac{1}{n+1}x^{n+1} + C$
$\int x^{-1} dx$	$=$	$\ln\|x\| + C$
$\int e^x dx$	$=$	$e^x + C$
$\int \sin(x) dx$	$=$	$-\cos(x) + C$
$\int \cos(x) dx$	$=$	$\sin(x) + C$
$\int \frac{dx}{\sqrt{1-x^2}} dx$	$=$	$\arcsin(x) + C = -\arccos(x) + C$
$\int \sec^2(x) dx$	$=$	$\tan(x) + C$
$\int \tan(x) dx$	$=$	$\ln\|\sec(x)\| + C$
$\int \csc^2(x) dx$	$=$	$-\cot(x) + C$
$\int \cot(x) dx$	$=$	$\ln\|\sin(x)\| + C$
$\int \frac{dx}{1+x^2} dx$	$=$	$\arctan(x) + C = -\text{arccot}(x) + C$
$\int \sec(x)\tan(x) dx$	$=$	$\sec(x) + C$
$\int \sec(x) dx$	$=$	$\ln\|\sec(x) + \tan(x)\| + C$
$\int \csc(x)\cot(x) dx$	$=$	$-\csc(x) + C$
$\int \csc(x) dx$	$=$	$-\ln\|\csc(x) + \cot(x)\| + C$
$\int \frac{dx}{\|x\|\sqrt{x^2-1}} dx$	$=$	$\text{arcsec}(x) + C = -\text{arccsc}(x) + C$

Trigonometric Identities

1	$=$	$\cos^2(x) + \sin^2(x)$
1	$=$	$\sec^2(x) - \tan^2(x)$
$\cos^2(x)$	$=$	$\frac{1}{2}[1 + \cos(2x)]$
$\sin^2(x)$	$=$	$\frac{1}{2}[1 - \cos(2x)]$
$\cos(a+b)$	$=$	$\cos(a)\cos(b) - \sin(a)\sin(b)$
$\sin(a+b)$	$=$	$\sin(a)\cos(b) + \cos(a)\sin(b)$

1. (50 points: Integration)

   (a) Find the antiderivative with constant of integration:

$$\int \sin(2x)e^{\cos(2x)}dx.$$

Solution. Let $u = \cos(2x)$, $du = -2\sin(2x)dx$, so that $\sin(2x)dx = -\frac{1}{2}du$. Then

$$\int \sin(2x)e^{\cos(2x)}dx = \int -\frac{1}{2}e^u du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{\cos(2x)} + C.$$

□

   (b) Find the antiderivative with constant of integration:

$$\int \cos^3(x)dx.$$

Solution. Note that $\cos^2(x) = 1 - \sin^2(x)$, so $\int \cos^3(x)dx = \int \cos(x)(1 - \sin^2(x))dx = \int \cos(x)dx - \int \sin^2(x)\cos(x)dx$. Letting $u = \sin(x), du = \cos(x)dx$, we get

$$\begin{aligned}\int \cos^3(x)dx &= \int \cos(x)dx - \int \sin^2(x)\cos(x)dx \\ &= \sin(x) - \int u^2 du \\ &= \sin(x) - \frac{u^3}{3} + C \\ &= \sin(x) - \frac{\sin^3(x)}{3} + C.\end{aligned}$$

□

(c) Find the antiderivative with constant of integration:

$$\int \frac{dx}{x^2+x}.$$

Solution.

$$\int \frac{1}{x^2+x}dx = \int \frac{1}{x(x+1)}dx = \int \left(\frac{A}{x} + \frac{B}{x+1}\right) dx$$
$$= \int \frac{A(x+1)+Bx}{x(x+1)}dx.$$

So $1 = A(x+1)+Bx$. If $x=0$, then $1 = A(1)+B(0) \implies A = 1$, and if $x=-1$, then $1 = A(0)+B(-1) \implies B=-1$, so we get

$$\int \frac{1}{x}dx - \int \frac{1}{x+1}dx = \ln|x| - \ln|x+1| + C.$$

□

(d) Find the number:

$$\int_{x=0}^{1} x(x^2+1)^7 dx.$$

Solution. Let $u = x^2+1, du = 2xdx$. Then $\int x(x^2+1)^7dx = \int \frac{1}{2}u^7du = \frac{u^8}{16} + C = \frac{x^2+1}{16} + C$. Then $\int_{x=0}^{1} x(x^2+1)^7dx = \frac{2^8}{16} - \frac{1}{16}$. □

(e) Find the number:

$$\int_{x=0}^{\frac{\pi}{2}} (2x+1)\sin(x)dx.$$

Solution. Let $u = 2x+1, dv = \sin(x)dx$, so $du = 2dx, v = -\cos(x)$. So by IBP,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} (2x+1)\sin(x)dx &= \left[-\cos(x)(2x+1) - \int -2\cos(x)dx\right]_0^{\frac{\pi}{2}} \\ &= \left[-\cos(x)(2x+1) + 2\sin(x)\right]_0^{\frac{\pi}{2}} \\ &= \left[-\cos\left(\frac{\pi}{2}\right)(\pi+1) + 2\sin\left(\frac{\pi}{2}\right)\right] \\ &- \left[-\cos(0)(1) + 2\sin(0)\right] = 3.\end{aligned}$$

□

(f) Find the number:

$$\int_{x=0}^{2} \left(1+x^2\right)^{-\frac{3}{2}} dx.$$

Solution. Let $x = \tan(\theta), dx = \sec^2(\theta)$. Then

$$\begin{aligned}\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx &= \int \frac{\sec^2(\theta)d\theta}{\sqrt{(1+\tan^2(\theta))^3}} = \int \frac{\sec^2(\theta)}{\sqrt{\sec^2(\theta)^3}} \\ &= \int \frac{\sec^2(\theta)}{\sec^3(\theta)}d\theta = \int \frac{1}{\sec(\theta)}d\theta = \int \cos(\theta)d\theta \\ &= \sin(\theta) + C = \sin(\arctan(x)) + C\end{aligned}$$

So

$$\int_{x=0}^{2} \left(1+x^2\right)^{-\frac{3}{2}} dx = \sin(\arctan(2)) - \sin(\arctan(0))$$

□

2. (50 points)
Consider the first quadrant region bounded by the curve

$$y = e^x$$

and the line

$$y = 1 + (e - 1)x.$$

This region runs from $x = 0$ to $x = 1$.
Consider also the solid obtained by revolving the given region about the y-axis. (It looks like a bowl).

(a) Write a definite integral(s) for the length of the boundary of the given region. (This is the sum of the lengths of the curve and the line). You do not need to evaluate the integral(s).

Solution. We have $f(x) = e^x, g(x) = 1+(e-1)x$, so $f'(x) = e^x, g'(x) = e - 1$. Then using the formula for arc length, we get

$$\int_0^1 \sqrt{1 + (e^x)^2}dx + \int_0^1 \sqrt{1 + (e - 1)^2}dx.$$

□

(b) Write a definite integral(s) for the surface area of the given solid. (This is the sum of the areas of the surfaces obtained by revolving the curve and by revolving the line). You do not need to evaluate the integral(s).

Solution. We are revolving about the y-axis, so we first rewrite the two curves as functions of y. $y = e^x \implies x = \ln(y)$, and $y = 1 + (e-1)x \implies x = \frac{y-1}{e-1}$. Letting $f(y) = \ln(x)$ and $g(y) = \frac{y-1}{e-1}$, we have $f'(y) = \frac{1}{y}$ and $g'(y) = \frac{1}{e-1}$. Note that between $x = 0$ and $x = 1$, y goes from 1 to e. Then using the formula for surface area, we get

$$\int_{y=1}^{e} 2\pi \ln(y)\sqrt{1 + \left(\frac{1}{y}\right)^2}\, dy + \int_{1}^{e} 2\pi \frac{y-1}{e-1}\sqrt{1 + \left(\frac{1}{e-1}\right)^2}\, dy$$

□

(c) Write a definite integral for the volume of the given solid. You do not need to evaluate the integral.

Solution. We use the shell method, with the line as the top function, and $y = e^x$ as the bottom function. The volume of the solid is

$$\int_0^1 2\pi x \left(1 + (e-1)x - e^x\right) dx.$$

□

3. (10 points: Extra Credit... you may skip this problem)
The functions f and g satisfy:

(a) $f''(x) = 3f(x)$ and $g(x) = 2g''(x)$,

(b) $f(5) = 3$ and $g(5) = \frac{1}{2}$,

(c) $f'(5) = g'(5) = 1$ and

(d) $f(0) = g(0) = f'(0) = g'(0) = 0$.

Find:

$$\int_{x=0}^{5} f(x)g(x)dx.$$

Solution. Note that $\int f(x)g(x)dx = \int f(x) \cdot 2g''(x)dx$. We apply IBP with $u_1 = f(x), dv_1 = 2g''(x)dx$, so $du_1 = f'(x)dx, v_1 = 2g'(x)$. Then

$$\int f(x)2g''(x)dx = f(x) \cdot 2g'(x) - \int f'(x) \cdot 2g'(x)dx.$$

We apply IBP again with $u_2 = f'(x), dv_2 = 2g'(x)dx$, and $du_2 = f''(x)dx, v_2 = 2g(x)dx$. Then

$$\begin{aligned}\int f(x)g(x)dx &= f(x) \cdot 2g'(x) - \left(f'(x) \cdot 2g(x) - \int 2g(x)f''(x)dx\right)\\ &= 2f(x)g'(x) - 2f'(x)g(x) + \int 2g(x) \cdot 3f(x)dx\\ &= 2f(x)g'(x) - 2f'(x)g(x) + 6\int f(x)g(x)dx.\end{aligned}$$

Then, we get

$$
\begin{aligned}
& -5\int f(x)g(x)dx = 2f(x)g'(x) - 2f'(x)g(x) \\
& \implies \int f(x)g(x)dx = \frac{1}{-5}\left(2f(x)g'(x) - 2f'(x)g(x)\right) + C
\end{aligned}
$$

So

$$
\begin{aligned}
\int_{x=0}^{5} f(x)g(x)dx &= \left[-\frac{1}{5}\left(2f(x)g'(x) - 2f'(x)g(x)\right)\right]_0^5 \\
&= \left[-\frac{1}{5}\left(2f(5)g'(5) - 2f'(5)g(5)\right)\right] - \left[-\frac{1}{5}\left(2f(0)g'(0) - 2f'(0)g(0)\right)\right] \\
&= \left[-\frac{1}{5}\left(2\cdot 3\cdot 1 - 2\cdot 1\cdot \frac{1}{2}\right)\right] - [0] \\
&= -\frac{1}{5}(6-1) = -1
\end{aligned}
$$

□