Friday, May 23, 2025 4:46 PM

Midterm III Practice Solutions (3ciii updated)

Problem 1

4m deep

Top (Rectangle) : 12m × 5m $\omega = 50\ N/m^3$

Side (triangle)

Part a $W = ?$

Coordinate: $y=0$ @ the bottom of the pool
$y=4$ @ the water surface

depth below surface $= 4-y$

Width (linear increase from $0 \to 4$) $= \frac{12}{4} y = 3y$

So Area $= 3y * 5 = 15y$

$$W = \int_a^b \omega \cdot A(y) \cdot \text{lifting distance}(y) \cdot dy$$

$$= \int_0^4 (50)(15y)(4-y)\,dy = 750\int_0^4 4y - y^2\,dy$$

$$= 750\left[2y^2 - \frac{1}{3}y^3\right]_0^4 = 750 \cdot \frac{32}{3} = \boxed{8000 J}$$

Part b $a=0$ $b=2$ (bott. $\to$ top of door)

depth = $4-y$ (since water surface is @ $y=4$)

$L(y) = 1\,m$ (door width)

$$F = \int_a^b w \cdot \text{depth}(y)\, L(y)\, dy$$

$$F = \int_0^2 (50)(4-y)(1)\, dy = 50\int_0^2 4-y\, dy$$

$$= 50\left[4y - \frac{1}{2}y^2\right]_0^2 = 50 \cdot 6 = \boxed{300\,N}$$

Problem 2

$$\frac{dy}{dt} = t^2 y \qquad y(0)=5 \quad y(3)=?$$

$$\int \frac{dy}{y} = \int t^2\, dt \Rightarrow \ln y = \frac{1}{3}t^3 + c$$

$$\Rightarrow e^{\ln y} = e^{\left[\frac{1}{3}t^3 + c\right]}$$

$$\Rightarrow y = e^{\frac{1}{3}t^3} \cdot e^c \qquad \text{let } e^c = A$$

some constant

$$y(t) = Ae^{\frac{1}{3}t^3}$$

$$y(0) = 5 = Ae^0 \Rightarrow A = 5$$

$$y(t) = 5e^{\frac{1}{3}t^3}$$

$$\boxed{y(3) = 5e^9}$$

or; $y = e^{\frac{1}{3}t^3} \cdot e^C \Rightarrow y(0) = e^0 \cdot e^C = e^C = 5$

$$\Rightarrow \ln e^C = \ln(5)$$

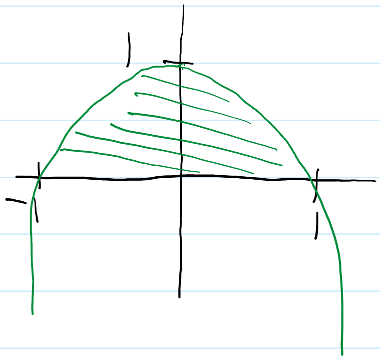
$$\Rightarrow \ln e^{c} = \ln(5)$$
$$c = \ln(5)$$

$$y(t) = e^{\ln(5)} e^{\frac{1}{3}t^3} = 5e^{\frac{1}{3}t^3}$$

$$y(3) = \boxed{5e^{9}}$$

<u>Problem 3</u>

$$y = 1 - x^2 \qquad x = -1 \qquad x = 1$$



(a) $A = \int_{-1}^{1} (1 - x^2)\, dx = \left[x - \frac{1}{3}x^3 \right]_{-1}^{1} = \boxed{\frac{4}{3}}$

(b) $\bar{x} = ? \qquad \bar{y} = ?$

$$\bar{x} = \frac{1}{M} \int_a^b \delta x \left[f(x) - g(x) \right] dx$$

$= \boxed{0}$ (bcuz the region is symmetric about y-axis)

$$\bar{y} = \frac{1}{M} \int_a^b \frac{1}{2} \delta \left[f^2(x) - g^2(x) \right] dx$$

$$= \frac{1}{2A}\int_a^b \left[f^2(x) - g^2(x)\right] dx$$

$$= \frac{\int_{-1}^{1} (1-x^2)^2 \, dx \quad (1)}{2\int_{-1}^{1} (1-x^2)\, dx \quad (2)}$$

(1) $\int_{-1}^{1} (1-x^2)^2 dx = \int_{-1}^{1} 1 - 2x^2 + x^4 \, dx$

$= \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1} = \frac{16}{15}$

Solved for in (a)

(2) $\int_{-1}^{1} (1-x^2) dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{3} * 2 = \frac{8}{3}$

$\bar{y} = \left[\frac{16}{15} * \frac{3}{8}\right] = \boxed{\frac{2}{5}}$

$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{2}{5}\right)}$

Part c) Revised

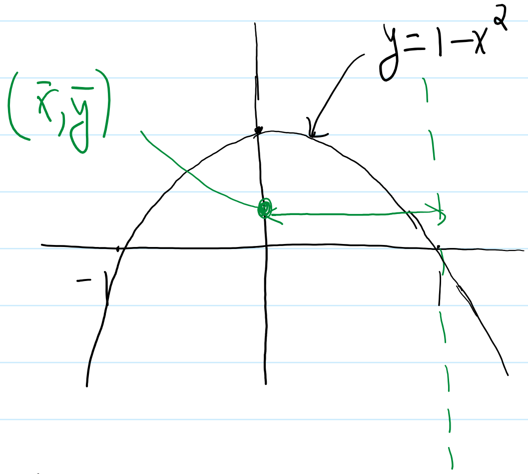
Solve using Pappus Theorem: $V = 2\pi A \rho$

where ρ = Distance from axis of revolution to centroid of the region.

(i) When revolving about the x-axis:

$$V = 2\pi A\bar{y} = 2\pi\left(\frac{4}{3}\right)\left(\frac{2}{5}\right) = \boxed{\frac{16\pi}{15}}$$

(ii) About line $x=1$



$\rho = 1-\bar{x}$

$$V = 2\pi\left(\frac{4}{3}\right)(1-\bar{x})$$

$$= 2\pi\left(\frac{4}{3}\right)(1) = \boxed{\frac{8\pi}{3}}$$

(iii) About line $y=1$

(same logic as above) $\Rightarrow \rho = 1-\bar{y}$

$$V = 2\pi\left(\frac{4}{3}\right)\left(1-\frac{2}{5}\right)$$

$$= 2\pi\left(\frac{4}{3}\right)\left(\frac{3}{5}\right) = \boxed{\frac{8\pi}{5}}$$

Problem 4

Part (a) $\int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_0^b e^{-x}\,dx$

$\Rightarrow \lim_{b\to\infty} -\left[e^{-x}\right]_0^b \Rightarrow \lim_{b\to\infty} -\left[e^{-b} - e^{0}\right]$

$\Rightarrow -\left[e^{-\infty} - 1\right] = \boxed{1}$

Part (b) $\int_0^{\infty} (1+\sin(x))\, e^{-x}\,dx$

* Use comparison Test

Side notes: if $f(x) \geq g(x) \geq 0$ on $[a, \infty)$ then,

1. if $\int_a^{\infty} f(x)\,dx$ converges then so does $\int_a^{\infty} g(x)\,dx$

2. if $\int_a^{\infty} g(x)\,dx$ diverges then so does $\int_a^{\infty} f(x)\,dx$

Solve: We know $-1 \leq \sin(x) \leq 1$

$+1 \quad +1 \quad +1$

$0 \leq 1+\sin(x) \leq 2$

$*e^{-x} \quad *e^{-x} \quad *e^{-x}$

$0 \leq [1+\sin(x)]e^{-x} \leq 2e^{-x}$

$$0 \le [1+\sin(x)]e^{-x} \le 2e^{-x}$$

Rewrite: $\underbrace{2e^{-x}}_{f(x)} \ge \underbrace{[1+\sin(x)]e^{-x}}_{g(x)} \ge 0$

We know that $f(x)$ converges since we solved for it in part (a)

Therefore $g(x)$ converges.

Conclusion: The integral converges by comparison to $2e^{-x}$

Problem 5 $\quad \frac{dy}{dx}$ @ $t=1$?

$$x(t) = te^{t} \qquad y(t) = t + \sin(\pi t)$$

$$\frac{dx}{dt} = t'e^{t} + t\cdot(e^{t})' = e^{t} + te^{t}$$

$$\frac{dy}{dt} = 1 + \pi\cos(\pi t)$$

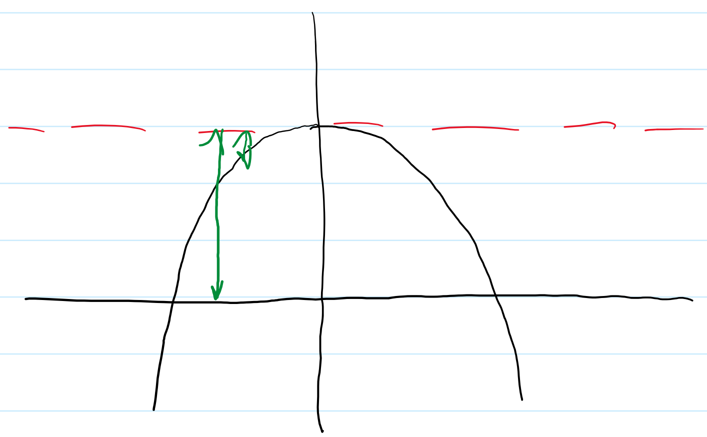
$$\frac{dy}{dt} * \frac{dt}{dx} = \frac{dy}{dx} \Rightarrow 1+\pi\cos(\pi t) * \frac{1}{e^{t}+te^{t}}$$

@ $t=1$; $\frac{dy}{dx} = \boxed{\frac{1+\pi\cos(\pi)}{2e}}$

End

Bonus: If solving Problem c(iii) using washer method, here is the correct solution:

About $y=1$



$R = 1 - 0 = 1$

$r = 1 - (1-x^2)$

$$V = \pi \int_{-1}^{1} (R)^2 - (r)^2 \, dx$$

$$= \pi \int_{-1}^{1} 1^2 - [1-(1-x^2)]^2 \, dx$$

$$= \pi \int^{1} 1 - (1-1+x^2)^2 \, dx$$

$$= \pi \int_{-1}^{1} 1 - (1 - 1 + x^2)^2 \, dx$$

$$= \pi \int_{-1}^{1} 1 - x^4 \, dx \Rightarrow \pi \left[x - \frac{1}{5} x^5 \right]_{-1}^{1}$$

$$\Rightarrow \boxed{V = \frac{8\pi}{5}}$$