Midterm III Solutions

Problem 1

$F = 10\text{ N} \quad x = 5\text{m} \quad F = kx \quad k = 10/5 = 2\text{ N/m}$

$$W = \int_0^5 2x\,dx = x^2\Big]_0^5 = \boxed{25 \text{ Jouls}}$$

Problem 2

$$\frac{dy}{dt} = \sqrt{y}\cdot\sqrt{t}$$

$$\frac{dy}{\sqrt{y}} = \sqrt{t}\,dt$$

$$y^{-1/2}\,dy = t^{1/2}\,dt$$

$$2\sqrt{y} = \frac{2}{3}t^{3/2} + c$$

$$\sqrt{y} = \frac{1}{3}t^{3/2} + \frac{1}{2}c$$

$$y = \left[\frac{1}{3}t^{3/2} + \frac{1}{2}c\right]^2$$

$$y = \left[\frac{1}{3} t + \frac{1}{2} C\right]$$

$$y(0) = 4 = \frac{1}{4} C^2 \Rightarrow C = 4$$

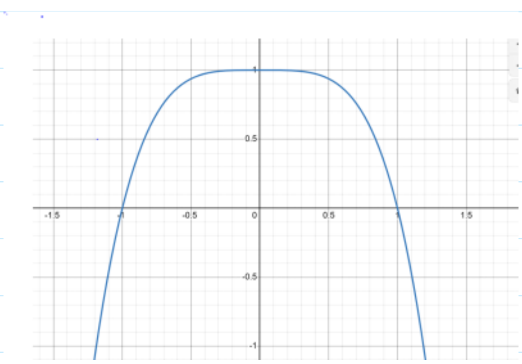
$$y(t) = \left(\frac{1}{3} t^{3/2} + 2\right)^2$$

$$y(9) = \left(\frac{1}{3} 9^{3/2} + 2\right)^2$$

$$= \left(\frac{27}{3} + 2\right)^2 = \boxed{121}$$

Problem 3

a)



$$A = \int_{-1}^{1} (1 - x^4)\, dx = \left[x - \frac{1}{5} x^5\right]_{-1}^{1}$$

$$= \left(1 - \frac{1}{5}\right) - \left(-1 + \frac{1}{5}\right)$$

$$= 1 - \frac{1}{5} + 1 - \frac{1}{5} = 2 - \frac{2}{5} = \boxed{\frac{8}{5}}$$

b) $\bar{x} = ?$ $\bar{y} = ?$

$$\bar{x} = \frac{1}{M}\int_a^b \delta x\,[f(x) - g(x)]\,dx$$

$$\bar{x} = \frac{\int_{-1}^{1} x\,(1-x^4)\,dx}{\frac{8}{5}} = \frac{5}{8}\cdot\int_{-1}^{1}(x - x^5)\,dx$$

$$= \frac{5}{8} * \left[\frac{x^2}{2} - \frac{x^6}{6}\right]_{-1}^{1}$$

$$= \frac{5}{8} * \left[\left(\frac{1}{2} - \frac{1}{6}\right) - \left(\frac{1}{2} - \frac{1}{6}\right)\right]$$

$$= \frac{5}{8} * 0 = \boxed{0}$$

$$\bar{y} = \frac{1}{M}\int_a^b \frac{1}{2}\delta\,[f^2(x) - g^2(x)]\,dx$$

$$\int_{-1}^{1}(1-x^4)^2\,dx \qquad \frac{5}{\ } \int^{1}(1 - 2x^4 + x^8)\,dx$$

$$= \frac{\int_{-1}^{1} (1-x^4)^2\, dx}{2\left(\frac{8}{5}\right)} = \frac{5}{16} * \int_{-1}^{1} \left(1 - 2x^4 + x^8\right) dx$$

$$\frac{5}{16} * \left[x - \frac{2}{5}x^5 + \frac{1}{9}x^9\right]_{-1}^{1}$$

$$\frac{5}{16} * \left[\left(1 - \frac{2}{5} + \frac{1}{9}\right) - \left(-1 + \frac{2}{5} - \frac{1}{9}\right)\right]$$

$$\frac{5}{16} * \frac{64}{45} = \boxed{\frac{4}{9}}$$

$\boxed{\text{Center of mass in } (0, 4/9)}$

Problem 4

$$\text{a)} \int_0^1 \frac{dx}{x^2} = \lim_{a \to 0^+} \int_a^1 x^{-2}\, dx = \left[-x^{-1}\right]_a^1$$

$$\lim_{a \to 0^+} \left(-1^{-1}\right) - \left(-a^{-1}\right)$$

$$\lim_{a \to 0^+} -1 + \frac{1}{a} \Rightarrow \text{Diverges}$$

Gets larger and larger ... → ∞

b) $$\frac{1}{x^2 \cos x} \geq \frac{1}{x^2} \quad \text{near } x \to 0^+$$

Since $\int \frac{1}{x^2}\,dx$ diverges, so does $\int \frac{1}{x^2 \cos x}\,dx$

Problem 5

$$x(t) = t + t^3 \qquad y(t) = t \ln t$$

at $t=1$; $\quad x(1) = 2 \quad y(1) = 0$

$$\frac{dx}{dt} = 1 + 3t^2 \qquad \frac{dy}{dt} = \ln t + 1$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\ln t + 1}{1 + 3t^2}$$

$$\left.\frac{dy}{dx}\right|_{t=1} = \text{Slope} = \frac{1}{4}$$

$$y - y_0 = m(x - x_0)$$

$$y - 0 = \frac{1}{4}(x - 2)$$

$$\boxed{y = \frac{1}{4}x - \frac{1}{2}}$$

Extra Credit

$y = x^{-k}$, x interval : $[1, \infty)$

a) $\int_1^\infty x^{-k}\,dx$ converges if $k > 1 \Rightarrow \boxed{k > 1}$

b) For revolution about x-axis,

$$V = \int_1^\infty \pi \left(x^{-k}\right)^2 dx$$

$= \pi \int_1^\infty x^{-2k}\,dx$ converges if $2k > 1 \Rightarrow \boxed{k > \frac{1}{2}}$

Asside: For values of k where

$A \Rightarrow$ diverges

$V \Rightarrow$ converges

Does the centroid of the region exist ?

$A \Rightarrow$ diverges if $k \leq 1$

$V \Rightarrow$ converges if $k > \frac{1}{2}$

So our k interval which satisfies these conditions is

$$\frac{1}{2} < k \leq 1$$

$$\bar{X} = \frac{1}{A}\int_1^{\infty} X \cdot X^{-k}\, dX = \frac{1}{A}\int_1^{\infty} x^{1-k}\, dX$$

Converges if $1-k < -1 \Rightarrow k > 2$

But we know that $\frac{1}{2} < k \leq 1$

Therefore this integral diverges.

$\bar{X} \Rightarrow$ undefined

The region does not have a centroid for the determined values of k.