Problem 6.1. *(Wi 04 9) Show that if some power of an n by n matrix A with field coefficients is* 0 *then* $A^n = 0$.

Problem 6.2. *(Fa 04 1) Assume that f and g are linear maps from a finite dimensional complex vector space V to itself. Show that there is some $v \in V - \{0\}$ for which $f(v)$ and $g(v)$ are collinear.*

Problem 6.3. *(Wi 06 3) Find all possible Jordan canonical forms for an n by n matrix M with $M^n = 0$.*

Problem 6.4. *(Fa 06 2) Find the dimension of a finite dimensional vector space V and all possible Jordan canonical forms for a linear operator from V to itself if the operator has characteristic polynomial $x^2(x-1)^4$ and minimal polynomial $x(x-1)^2$.*

Problem 6.5. *(Wi 07 6)*

(1) *Give an example of a* 4 *by* 4 *matrix with* 3 *as the only eigenvalue and a two dimensional eigenspace.*
(2) *Write K for the set of all matrices satisfying the above conditions and ϕ for the action of $GL_4(\mathbb{C})$ on K by conjugation (so if $A \in GL_4(\mathbb{C})$ and $M \in K$ then $\phi_A(M) = AMA^{-1}$). How many orbits does this action have?*

Problem 6.6. *(Wi 07 6) Show that if A is a finite dimensional square complex matrix and e^A is defined to be $1 + A + \frac{A^2}{2!} + \ldots + \frac{A^k}{k!} + \ldots$ then $\det(e^A) = e^{tr(A)}$.*

Problem 6.7. *(Fa 08 3) Assume that M is a* 3 *by* 3 *complex matrix with characteristic polynomial $x^3 + 5x^2 + 3x + (9 - i)$.*

(1) *Find the determinant of M^2.*
(2) *Find the trace of M^2.*
(3) *Find the characteristic polynomial of M^2.*

Problem 6.8. *(Fa 09 3) Show that if A and B are linear transformations on a finite dimensional vector space then* $\dim(\ker(AB)) \leq \dim(\ker(A)) + \dim(\ker(B))$.

Problem 6.9. *(Sp 11 1) Let M be an n by n complex matrix. Show that the following three conditions are equivalent:*

(1) $MM^* = M^*M$ *(that is M is normal)*
(2) $M = A + iB$ *for some A and B commuting and both self-adjoint.*
(3) $M = RU$ *for some R and U commuting with R self-adjoint and U unitary.*

Problem 6.10. *(Sp 12 1) Show that every real, square, upper triangular matrix which commutes with its transpose must be diagonal.*

6.1. **Demonstration Problems.**

Problem 6.11. *(Wi 04 12) A matrix M over $\mathbb{C}$ has characteristic polynomial $(x-3)^4(x+4)^3$ a one dimensional eigenspace for eigenvector -4 and a two dimensional eigenspace for eigenvector* 3.

(1) *Find the determinant of M.*
(2) *Find the trace of M^2.*
(3) *Find all possible Jordan canonical forms for M.*

Problem 6.12. *(Wi 08 2) Is every n by n complex matrix similar to its transpose?*

Problem 6.13. *(Wi 09 1)*

(1) *Find a complex matrix M with*
$$M^2 = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$$
.
(2) *How many such matrices are there. (Note that if M is a solution so is $-M$.)*