1. Infinite Groups F14.1

Fall 2014 Problem 1. Let G_1, G_2 be finite index subgroups of a group G. Show that the intersection $G_1 \cap G_2$ also has finite index in G.

1.1. Ideas F14.1.

- Consider the action of G on some set such as:
- (a) conjugates of a subgroup
- (b) cosets of a subgroup
- If the set is finite then so is the index of the subgroup that acts trivially (at most the order of the symmetric group which is the factorial of the set size).
- (b)(i) This looks better here: The number of cosets is the index of the subgroup.
- (b)(ii) Every element of G acting trivially on both the cosets of G_1 and G_2 is in the intersection.
- (b)(iii) Two actions on sets can be combined into one action on the product.

1.2. Write up F14.1.

Consider the group homomorphism $f : G \to S$ where $S = S_1 \times S_2$ and S_i is the symmetric group on the left cosets hG_i of G_i in G given by $f(g)(h_1G_1, h_2G_2) = (gh_1G_1, gh_2G_2)$.

Claim:

(1) f is (as claimed) a group homomorphism.
(2) If K is the kernel of f then $K \subseteq G_1 \cap G_2$.
(3) The order of S is finite.

By (1) and (3) K is finite index in G and hence by (2) so is $G_1 \cap G_2$.

Justifications:

(1) This is an easy direct check.
(2) The intersection $G_1 \cap G_2$ is the stabilizer of the point $(eG_1, eG_2) \in S$. The kernel K is the intersection of all stabilizers and hence a subset.
(3) From the definition of S one has $|S| = I_1!I_2!$ with I_i the index of G_i in G and hence finite by hypothesis.

2. Infinite Groups S16.3

Spring 2016 Problem 3. Let G be a group generated by elements a, b each of which has order 2. Prove that G contains a subgroup of index 2.

2.1. Ideas S16.3.

- Look for a map to a group of order 2.
- Use the surjection from F_2 factoring through the infinite dihedral group.
- Look for a subgroup directly such as $\langle ab, ba\rangle$.

2.2. Write up S16.3.

Write $F_2/\langle A^2, B^2\rangle = D_2 = \{\prod_{n=1}^{\ell} C_n | C_n \in \{A, B\}, C_n \neq C_{n+1}\}$ for the dihedral quotient of the free group and $E_2 = \{\prod_{n=1}^{2\ell} C_n | C_n \in \{A, B\}, C_n \neq C_{n+1}\}$ for the even subgroup of D_2. Write $\alpha : D_2 \to G$ for the group homomorphism with $\alpha(A) = a$ and $\alpha(B) = b$ and $H = \alpha(E_2)$.

Claim:

(1) $\alpha : D_2 \to G$ is a surjective homomorphism.
(2) $a \notin H$

By (1) $G = H \cup aH$ and by (2) $G \neq H$ so H has index 2.

Justifications:

(1) This is immediate from the hypotheses on G.
(2) If $a = h = \prod^{2\ell} c_i \in H$ then either $c_1 = a$ or $c_{2\ell} = a$. In the first case $b = baa = bah = \prod^{2\ell-2} c_i$. In the second case $b = aab = hab = \prod^{2\ell-2} c_i$. Repeating ℓ times gives $a = e$ or $b = e$ contradicting that they have order 2.