1. Linear Algebra F16.3

Fall 2016 Problem 3: Show that every linear map $A : \mathbb{R} \to \mathbb{R}$ has both a 1-dimensional invariant subspace and a 2-dimensional invariant subspace.

1.1. Ideas F16.3.

- Jordan form should help but this may involve $\mathbb{C}$.
- An eigenvector gives a 1-dimensional invariant subspace.
- Rational canonical form works.
- The classification of finitely generated modules over a pid fits here with $\mathbb{R}^3$ the module, $\mathbb{R}[x]$ the pid and $p(x)m = p(A)m$. Under this dictionary a submodule is the same as an invariant subspace.

1.2. Write up F16.3.

Theorem: Every finitely generated module over a pid R is isomorphic to a direct sum of modules $R/(r)$.

Theorem: If T is a pid so is $T[x]$.

Theorem: Every field is a pid.

Thus $\mathbb{R}[x]$ is a pid.

Theorem: Every $p(x) \in \mathbb{R}[x]$ factors into linear and quadratic terms.

Note that the $\mathbb{R}$-vector space dimension of an $\mathbb{R}[x]$-module $\mathbb{R}[x]/(p) \oplus \mathbb{R}[x]/(q)$ is the degree of p plus the degree of q.

Theorem:(CRT) If $(p, q) = 1$ then $\mathbb{R}[x]/(pq) \cong \mathbb{R}[x]/(p) \oplus \mathbb{R}[x]/(q)$.

Combining these an $\mathbb{R}[x]$-modue which is a 3-dimensional $\mathbb{R}$-vector space is isomorphic to one of the following:

1. $\mathbb{R}[x]/(p^3)$ with p linear,
2. $\mathbb{R}[x]/(p) \oplus \mathbb{R}[x]/(q)$ with p linear and q quadratic,
3. $\mathbb{R}[x]/(p) \oplus \mathbb{R}[x]/(q) \oplus \mathbb{R}[x]/(r)$ with p, q and r linear.

In the latter two cases the summands are invariant subspaces. In the first case $p\mathbb{R}[x]/(p^3)$ and $p^2\mathbb{R}[x]/(p^3)$ are 2-dimensional and 1-dimensional invariant subspaces respectively.