# 1. Tensors F18.4

**Fall 2018 Problem 4.** Solve the following:

(a) Prove that $R/I \otimes_R R/J \cong R/(I+J)$ for $R$ a commutative ring, and $I, J \subset R$ are ideals.
(b) Find the dimension of $\mathbb{Q}[x,y]/(x^2+y^2) \otimes_{\mathbb{Q}[x,y]} \mathbb{Q}[x,y]/(x+y^3)$ as a vector space over $\mathbb{Q}$, or explain why it is infinite.

## 1.1. Ideas F18.4.

(a) Try the case $R = \mathbb{Z}$.
(a) Try the universal property of $\otimes$ for the map in one direction.
(a) $R^2/(RI+RJ)$ seems better. If $1 \in R$ they are the same.
(a) Try the construction of $\otimes$ for the map the other direction.
(b) Eliminate the variable $x$ from $\mathbb{Q}[x,y]/I$ if $x - p(y) \in I$.

## 1.2. Write up F18.4. 1

(a) Write $T = R^2/(RI+RJ) \in R$-Mod and prove that $R/I \otimes_R R/J \cong T$ as $R$-modules. If $1 \in R$ then $T = R/(I+J)$.

Recall that if $M$ and $N$ are $R$-modules then $M \otimes_R N$ has

(1) the universal property that there is an $R$-bilinear map $j : M \times N \to M \otimes_R N$ and any $R$-bilinear map $f : M \times N \to W$ factors uniquely through $j$.
(2) the construction $M \otimes_R N = R^{\oplus(M\times N)}/\sim$ with $j((m,n)) = m \otimes n$.

Consider $f : R/I \times R/J \to R^2/(RI+RJ)$ given by $f((r+I, r'+J)) = (rr' + (RI+RJ))$.

It is easy to check that $f$ is well defined and $R$-bilinear so by (1) there is $g : R/I \otimes_R R/J \to R^2/(RI+RJ)$ with $g \circ j = f$.

Consider $h : R^{\oplus(M\times N)} \to R^2/(RI+RJ)$ given by $h(r \otimes r') = rr' + (RI+RJ)$.

It is easy to check that $h$ induces a map $k$ from the tensor product and $k = g^{-1}$ proving part (a).

(b) From (a) one has the desired module isomorphic to $\mathbb{Q}[x,y]/(x^2+y^2, x+y^3) = \mathbb{Q}[y]/(-y^6+y^2)$ which has a $\mathbb{Q}$-vector space basis $\langle 1, y, y^2, y^3, y^4, y^5\rangle$ and is of dimension 6.

# 2. Tensors F16.6

**Fall 2016 Problem 6.** Show that the $\mathbb{R}$-modules $L = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $L = \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are not isomorphic.

## 2.1. Ideas F16.6.

- Consider the $\mathbb{R}$- dimensions.

2.2. **Write up F16.6.** As $\mathbb{R}$- vector spaces the first is 4 dimensional and the second is 2 dimensional.