Fall 2014 Problem 5. Let ρ be a representation of a finite group G on a vector space V and let $v \in V$.

(a) Show that averaging $\rho_g(v)$ over G gives a vector $\overline{v} \in V$ which is fixed by G.
(b) What can you say about this vector when ρ is an irreducible representation?

Fall 2012 Problem 6. Consider the dihedral group $D_4 = \langle r, s : s^2 = r^4 = 1, rs = sr^{-1} \rangle$ of order 8.

(a) Find the conjugacy classes of D_4.
(b) Find the character table of D_4.

Fall 2006 Problem 5: Define $\phi_A(X) = AXA^{-1}$ if $X \in \mathbb{C}^{2\times 2}$ and $A \in GL_2(\mathbb{C})$.

(1) Check that this defines a linear representation of $GL_2(\mathbb{C})$
(2) Show that this representation is reducible.
(3) Write down one element of each orbit of the above action and in each case find the stabilizer of that element.

Winter 2007 Problem 3: Consider the following 5 by 5 character table for the finite group G in which a is a primitive 5th root of unity:

1	1	1	1	1
3	-1	0	a^3+a^2+1	a^4+a+1
3	-1	0	a^4+a+1	a^3+a^2+1
4	0	1	-1	-1
5	1	-1	0	0

(1) Show that G is simple and of order 60 and find the orders of its conjugacy classes.

(2) Find the decomposition into irreducible representations of the tensor produt of the two 3-dimensional irreducible representations.

Spring 2013 Problem 3. The dihedral group D_4 acts as the symmetries of a square in the plane $\mathbb{R}^2$ with coordinates x and y. Suppose that the corners of this square are at $(\pm 1, \pm 1)$. Then D_4 acts linearly and it therefore has an induced action on the vector space V_n of homogeneous polynomials in x and y of degree n. Find the character of V_n viewed as a representation of D_4. (Note: the character will depend on n.)

Fall 2007 Problem 1: Show that every matrix in the image of a complex representation of a finite group is diagonalizable.

Fall 2018 Problem 6. Let V be the subspace of $\mathbb{C}^3$ spanned by $v_1, v_2 = (1,-1,0), (0,1,-1)$ which is an invariant subspace under the permutation action of S_3, and so gives a two dimensional representation $\rho : S_3 \to \mathrm{GL}(V)$.

(a) Write down the matrices of $\rho(\sigma)$ in this basis.
(b) Describe a Hermitian inner product $\langle , \rangle$ on V in the basis v_1, v_2 which is G-invariant. i.e. $\langle \rho(g)u, \rho(g)v\rangle = \langle v, u\rangle$.
(c) Describe the tensor product $V \otimes_R V$, as an R-module where $R = \mathbb{C}[H]$ and $H \subset S_3$ is the subgroup $\{1, (123), (132)\}$.

Spring 2015 Problem 6. Let G be a finite group and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ a representation.

(a) Show $\delta : G \to \mathbb{C}$, $g \mapsto \det(\rho(g))$ is linear character of G (i.e. a group homomorphism to the multiplicative group).
(b) Show: If $\delta(g) = -1$ for some $g \in G$ then G has a normal subgroup of index 2.
(c) Show: If G has order $2k$, k odd, then G has a normal subgroupu of index 2.
(d) Let $\chi(g) = \mathrm{tr}(\rho(g))$ and $g \in G$ an involution. Show: (i) $\chi(g)$ is an integer; (ii) $\chi(g) \equiv \chi(1) \bmod 2$; (iii) if G has no normal subgroup of index 2, then $\chi(g) \equiv \chi(1) \bmod 4$.

Spring 2014 Problem 5: Let V be a linear representation of a group G over the field $\mathbb{Q}(\sqrt{2})$ with character $\chi : G \to \mathbb{Q}(\sqrt{2})$. Consider V as a vector space $V_{\mathbb{Q}}$ over the subfield $\mathbb{Q}$ with the same action of G. Find the character $\chi_{\mathbb{Q}}$ of this new representation in terms of the original character χ.