MAT 249, 2021, Day 1, Finite Groups

(1) **Fall 2020 Problem 4:**
   (a) Let G be a finite group, and let H be a normal subgroup of G. Let P be a p-Sylow subgroup of H, and let K be the normalizer of P in G. Establish the equality $G = HK$.
   (b) Let G be a group of order 120, and let $H \subset G$ be a subgroup of order 24. Suppose that there is at least one left coset of H in G (other than H itself) that is also a right coset of H in G. Prove that H is a normal subgroup of G.

(2) **Fall 2019 Problem 1:** Let $G = GL_3(\mathbb{F}_3)$ be the general inear group of 3×3 matrices over a finite field $\mathbb{F}_3$.
   (a) Show that there is no element $x \in G$ with order 27. (This corrects an error in the original.)
   (b) How many subgroups of order 27 does G have?

(3) **Fall 2017 Problem 6:** Prove that if p is a prime number, then every group G with p^2 elements is abelian.

(4) **Fall 2016 Problem 2:** Let G be a finite group that acts transitively on a set X of cardinality ≥ 2. Show that there exists an element $g \in G$ which acts on X without any fixed points. Is the same true if G is infinite?

(5) **Fall 2015 Problem 1:** Let G be a finite group such that all Sylow subgroups of G are normal and abelian. Show that G is abelian.

(6) **Fall 2015 Problem 2:** For a finite group G define the subset $G^2 = \{g^2 : g \in G\}$. Is it true that G^2 is always a subgroup?

(7) **Fall 2014 Problem 2:** Show that every subgroup of index equal to the smallest prime dividing the order of a group is normal.

(8) **Fall 2013 Problem 4:** Let G be a group with an odd number of elements that has a normal subgroup N with 17 elements. Show that N lies in the center of G.