MAT 249, 2021, Day 2, Tensors

(1) **Fall 2020 Problem 3.** Let R be a commutative ring with unit, and let M, N be R-modules whose underlying set is finite. Prove that $M \otimes_R N$ is also a finite set.

(2) **Fall 2019 Problem 4.**
   (a) If R is a commutative ring with unit, and M, N are R-modules, how does one describe the R-module structure of $M \otimes_R N$?
   (b) Let M be the module over $R = \mathbb{Z}[x]$ whose total space is $\mathbb{Z}$, such that x acts by multipliction by -1. Compute the tensor product $M \otimes_R M$ as an R-module.

(3) **Fall 2018 Problem 2.** Let $\mathbb{H}$ be the real quarternions. Then $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}$ is isomorphic to which of the following rings? Prove your answer.
   (a) $\mathbb{C} \times \mathbb{C}$
   (b) $\mathbb{C} \times \mathbb{C} \times \mathbb{C} \times \mathbb{C}$
   (c) $\mathcal{M}_2(\mathbb{C})$
   (d) $\mathcal{M}_2(\mathbb{R})$
   (e) $\mathcal{M}_2(\mathbb{H})$
   (f) $\mathcal{M}_2(\mathbb{R}) \times \mathcal{M}_2(\mathbb{R})$

(4) **Fall 2018 Problem 4.** Solve the following:
   (a) Prove that $R/I \otimes_R R/J \cong R/(I + J)$ for R a commutative ring, and $I, J \subset R$ are ideals.
   (b) Find the dimension of $\mathbb{Q}[x, y]/(x^2 + y^2) \otimes_{\mathbb{Q}[x,y]} \mathbb{Q}[x, y]/(x + y^3)$ as a vector space over $\mathbb{Q}$, or explain why it is infinite.

(5) **Fall 2016 Problem 6.** Show that the $\mathbb{R}$-modules $L = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $M = \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are not isomorphic.

(6) **Fall 2014 Problem 6.** If R is a commutative ring with identity, and S a multiplicative subset, then every ideal J of $S^{-1}R$ is of the form $S^{-1}I$ for some ideal I of R. Is I uniquely determined by J? Why or why not?

(7) **Fall 2013 Problem 2.** Consider an attempt to make an $\mathbb{R}$-linear map

$$f : \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$$

or

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C},$$

in either direction given be the formula $f(x \otimes y) = x \otimes y$. In which direction is this map well defiined? Is it then surjective? Is it injective?

(8) **Winter 2009 Problem 4:** Let P be any set of primes in $\mathbb{Z}$. Find a commutative ring R containing $\mathbb{Z}$ for which the irreducible elements are precisely units times elements of P.
(Hint: consider $\mathbb{Z}[\frac{1}{2}] = \{\frac{a}{2^r}\} \subseteq \mathbb{Q}$).

(9) **Fall 2009 Problem 5:** Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \oplus \mathbb{C}$ are isomorphic as $\mathbb{C}$-algebras.

(10) **Winter 2008 Problem 5:** What are the possible orders of the tensor product as $\mathbb{Z}$ modules of two Abelian groups which each have order 25?